

(WEAK) m -EXTREMALS AND m -GEODESICS

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ABSTRACT. We present a collection of results on (weak) m -extremals and m -geodesics, concerning general properties, the planar case, quasi-balanced pseudoconvex domains, complex ellipsoids, the Euclidean ball and boundary properties. We prove 3-geodesity of 3-extremals in the Euclidean ball. Equivalence of weak m -extremality and m -extremality in some class of convex complex ellipsoids, containing symmetric ones and C^2 -smooth ones is showed. Moreover, first examples of 3-extremals being not 3-geodesics in convex domains are given.

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1. INTRODUCTION

1.1. Idea of (weak) m -extremals and m -geodesics. This paper may be treated as a continuation of [18], where these objects were investigated from the point of view of geometric function theory. The notion of m -extremals comes from [1] (cf. [2]) and was used to studying interpolation problems in the symmetrised bidisc — a special domain appearing in what is known as μ -synthesis. It is a kind of approach to the spectral Nevanlinna-Pick problem (see also [17]), in which domains like the tetrablock and the pentablock occur naturally. They have been intensively studied of late in geometric function theory. However, m -extremals are in some sense too restricted. Therefore, it was natural to define *weak* m -extremals; on the other side, a stronger notion of m -geodesics let us produce m -extremals efficiently (Ł. Kosiński and W. Zwonek introduced both notions).

G. Pick [26] was the first who observed that Blaschke products have some extremal property in the unit disc \mathbb{D} . The result formulated in our language claims that a holomorphic function $f : \mathbb{D} \rightarrow \mathbb{D}$ is a (weak) m -extremal if and only if it is a non-constant Blaschke product of degree at most $m - 1$. More famous Pick (or

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Nevanlinna-Pick) theorem [22, 23, 24] describes situations, in which a given interpolation problem in \mathbb{D} has a solution. These results were obtained by the Schur's reduction [9, 30].

A more general view on extremal problems (using special functionals) was presented in [27]. A. Edigarian developed these ideas in the crucial work [6], where among others the necessary form of weak m -extremals in complex ellipsoids is given. We will strongly use that result. A related problem with infinitely many interpolation data was studied in [3].

There is a significant relationship between discussed objects and the theory of holomorphically contractible functions [12, 13, 15] — weak m -extremals (resp. m -geodesics) generalize classical Lempert extremals (resp. geodesics).

1.2. Notation and definitions. In what follows and if not mentioned otherwise, we assume that $m \geq 2$ is natural. Let $D \subset \mathbb{C}^n$ be a domain. Denote by $\mathcal{O}(\overline{\mathbb{D}}, D)$ the set of mappings holomorphic in a neighborhood of $\overline{\mathbb{D}}$ with values in D . Moreover, let $\lambda_1, \dots, \lambda_m \in \mathbb{D}$ be distinct points (distinct = pairwise distinct).

A holomorphic mapping $f : \mathbb{D} \rightarrow D$ is called a *weak m -extremal* for $\lambda_1, \dots, \lambda_m$ if there is no map $h \in \mathcal{O}(\overline{\mathbb{D}}, D)$ such that $h(\lambda_j) = f(\lambda_j)$, $j = 1, \dots, m$. Naturally, weak m -extremality means weak m -extremality for some $\lambda_1, \dots, \lambda_m$.

If the above condition is satisfied for any different numbers $\lambda_1, \dots, \lambda_m \in \mathbb{D}$, we say that f is an *m -extremal*.

Note that a map $f \in \mathcal{O}(\mathbb{D}, D)$ is a weak m -extremal for $\lambda_1, \dots, \lambda_m$ if and only if there is no $g \in \mathcal{O}(\mathbb{D}, D)$ with $g(\lambda_j) = f(\lambda_j)$, $j = 1, \dots, m$, and $g(\mathbb{D}) \subset\subset D$ (cf. Lemma 2.1(a)).

For $\alpha \in \mathbb{D}$ define the *Möbius function*

$$m_\alpha(\lambda) := \frac{\lambda - \alpha}{1 - \overline{\alpha}\lambda}, \quad \lambda \in \mathbb{D}.$$

We shall consider finite *Blaschke products*, that is functions

$$B := \zeta \prod_{j=1}^k m_{\alpha_j},$$

where $k \in \mathbb{N}_0$, $\alpha_j \in \mathbb{D}$, $\zeta \in \mathbb{T} := \partial\mathbb{D}$ (we assume that $0 \notin \mathbb{N}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$). The number k is said to be a *degree* and is denoted by $\deg B$. In case $k = 0$, the function B is a unimodular constant ζ .

A holomorphic mapping $f : \mathbb{D} \rightarrow D$ is said to be an *m -geodesic* if there exists $F \in \mathcal{O}(D, \mathbb{D})$ such that $F \circ f$ is a non-constant Blaschke product of degree at most $m - 1$. We call such F an *m -left inverse*.

Note that a holomorphic map is a weak 2-extremal (resp. a 2-geodesic) if and only if it is a Lempert extremal (resp. a geodesic). Recall that a mapping $f \in \mathcal{O}(\mathbb{D}, D)$ is a *Lempert extremal* if $\ell_D(f(\lambda_1), f(\lambda_2)) = \mathbf{p}(\lambda_1, \lambda_2)$ for some different $\lambda_1, \lambda_2 \in \mathbb{D}$, where \mathbf{p} stands for the Poincaré distance on \mathbb{D} and

$$\ell_D(z, w) := \inf\{\mathbf{p}(\lambda_1, \lambda_2) : \lambda_1, \lambda_2 \in \mathbb{D} \text{ and } \exists f \in \mathcal{O}(\mathbb{D}, D) : f(\lambda_1) = z, f(\lambda_2) = w\}$$

is the *Lempert function* of D . We call f a *geodesic* if $\mathbf{c}_D(f(\lambda_1), f(\lambda_2)) = \mathbf{p}(\lambda_1, \lambda_2)$ for any (equivalently for some different) $\lambda_1, \lambda_2 \in \mathbb{D}$, where

$$\mathbf{c}_D(z, w) := \sup\{\mathbf{p}(F(z), F(w)) : F \in \mathcal{O}(D, \mathbb{D})\}$$

is the *Carathéodory pseudodistance* of D . This is exactly the case, when f has a 2-left inverse.

From the description of m -extremals in \mathbb{D} it follows that in any domain m -geodesicity implies m -extremality. It is obvious that for all considered notions the 'level' m implies $m+1$. They are invariant under biholomorphisms and compositions with automorphisms of \mathbb{D} .

1.3. Main results. It is known from [18] that in the Euclidean ball we have m -extremals being not m -geodesics for $m \geq 4$. The missing case is solved in Theorem 5.8: **any 3-extremal of \mathbb{B}_n is a 3-geodesic.**

By the Lempert theorem [19, 20] (cf. [13, Chapter 11] and [28]), any weak 2-extremal of a convex domain is a 2-geodesic, in particular a 2-extremal. Thus the following question about a ‘weak’ generalization of this result seems to be important: whether a weak m -extremal, $m \geq 3$, of a convex domain has to be an m -extremal. We do not know it, however we have found **convex domains, in which for any $m \geq 3$ there exists an m -extremal being not an m -geodesic** (first convex examples for $m = 3$). One of them is the complex ellipsoid $\mathcal{E}(1/2, 1/2)$, where $\mathcal{E}(p) := \{z \in \mathbb{C}^n : |z_1|^{2p_1} + \dots + |z_n|^{2p_n} < 1\}$ (Proposition 4.2). Another example follows from Proposition 4.4.

The next results, we would like to draw attention to, are Propositions 4.12 and 4.13. We define some family $\mathcal{E}(p)$, $p \in \mathcal{S}_n$, which contains all symmetric convex and all \mathcal{C}^2 -smooth complex ellipsoids. It turns out that **in $\mathcal{E}(p)$ such that $p \in \mathcal{S}_n$, weak m -extremality equals m -extremality.** We get moreover some l -extremality of all weak m -extremals for p such that $p_j/q_j \in \mathbb{N}$, $j = 1, \dots, n$, where $q \in \mathcal{S}_n$ (l is bounded by a function of m and p).

We also deal with dividing m -geodesics of quasi-balanced pseudoconvex domains by the identity function on the unit disc. The aim is to decide whether the new map is an $(m-1)$ -geodesic. The reasoning used in the proof of [7, Theorem 3] gives a positive answer for $m = 3$. Most interesting is the balanced case, in which we give convex counterexamples for $m \geq 4$ (Corollary 4.3 and Propositions 4.4, 4.5).

Some of the results answer partially to questions posed at the end. Their occurrences in the text are marked (Pn).

We have already two general questions: whether it is possible to find a 2-extremal being not a 2-geodesic (P1), and whether there exists an m -extremal, which is not any k -geodesic (P2).

2. GENERAL PROPERTIES AND THE PLANAR CASE

Denote by $|\cdot|$ the Euclidean norm and $\|f\|_S := \sup_S |f|$.

Lemma 2.1. *Let $D \subset \mathbb{C}^n$ be a domain and let $\lambda_1, \dots, \lambda_m \in \mathbb{D}$ be different points.*

- (a) *Fix $z_1, \dots, z_m \in D$. Then there exists $h \in \mathcal{O}(\mathbb{D}, D)$ such that $h(\lambda_j) = z_j$, $j = 1, \dots, m$, if and only if there exists $g \in \mathcal{O}(\mathbb{D}, D)$ such that $g(\lambda_j) = z_j$, $j = 1, \dots, m$, and $g(\mathbb{D}) \subset\subset D$.*
- (b) *Let $\mathbb{D} \ni \lambda_j^{(k)} \rightarrow \lambda_j$, $k \rightarrow \infty$, and let $f : \mathbb{D} \rightarrow D$ be a weak m -extremal for $\lambda_1^{(k)}, \dots, \lambda_m^{(k)}$. Then f is a weak m -extremal for $\lambda_1, \dots, \lambda_m$.*
- (c) *Assume that $f_k, f \in \mathcal{O}(\mathbb{D}, D)$, $f_k(\lambda_j) \rightarrow f(\lambda_j)$, $k \rightarrow \infty$, and f_k are weak m -extremals for $\lambda_1, \dots, \lambda_m$. Then f is, as well.*
- (d) *If $\mathcal{O}(\mathbb{D}, D) \ni f_k \rightarrow f \in \mathcal{O}(\mathbb{D}, D)$ pointwise and any f_k is an m -extremal, then f is, as well.*

Proof. Let $w_1, \dots, w_m \in \mathbb{C}^n$, $w := (w_1, \dots, w_m)$. The polynomial mapping

$$P_w(\lambda) := \sum_{j=1}^m \left(\prod_{k \neq j} \frac{\lambda - \lambda_k}{\lambda_j - \lambda_k} \right) w_j, \quad \lambda \in \mathbb{C},$$

has the property that $P_w(\lambda_l) = w_l$, $l = 1, \dots, m$, and $\|P_w\|_S \rightarrow 0$ if $w \rightarrow 0$, for any $\emptyset \neq S \subset\subset \mathbb{C}$.

- (a) If we have g , then consider $g_r(\lambda) := g(\lambda/r)$, $\lambda \in r\mathbb{D}$, $r > 1$. As

$$g_r(\lambda_j) + g(\lambda_j) - g_r(\lambda_j) = z_j,$$

we put $w_j = w_j(r) := g(\lambda_j) - g_r(\lambda_j)$ and $h := g_r + P_{w(r)}$ for r close enough to 1.

(b) Suppose that there exists $h \in \mathcal{O}(\mathbb{D}, D)$ such that $h(\lambda_j) = f(\lambda_j)$, $j = 1, \dots, m$, and $h(\mathbb{D}) \subset \subset D$. We proceed similarly as above with the equation

$$h(\lambda_j^{(k)}) + h(\lambda_j) - h(\lambda_j^{(k)}) + f(\lambda_j^{(k)}) - f(\lambda_j) = f(\lambda_j^{(k)})$$

and get a contradiction with weak m -extremality of f for $\lambda_1^{(k)}, \dots, \lambda_m^{(k)}$ if $k \gg 1$.

(c) If there were exist $h \in \mathcal{O}(\mathbb{D}, D)$ with $h(\lambda_j) = f(\lambda_j)$, $j = 1, \dots, m$, and $h(\mathbb{D}) \subset \subset D$, we would have

$$h(\lambda_j) + f_k(\lambda_j) - f(\lambda_j) = f_k(\lambda_j),$$

whence for big k the map f_k would be not a weak m -extremal for $\lambda_1, \dots, \lambda_m$.

(d) It follows from (c). \square

From the definition of a weak m -extremal follows

Lemma 2.2. *Let $D_j \subset \mathbb{C}^{k_j}$ be domains and let $f_j \in \mathcal{O}(\mathbb{D}, D_j)$, $j = 1, \dots, n$. Then the mapping $(f_1, \dots, f_n) : \mathbb{D} \rightarrow D_1 \times \dots \times D_n$ is a weak m -extremal for $\lambda_1, \dots, \lambda_m$ if and only if at least one of the maps f_j is a weak m -extremal for $\lambda_1, \dots, \lambda_m$.*

In particular, we have the following description for the polydisc \mathbb{D}^n .

Remark 2.3. Let $f : \mathbb{D} \rightarrow \mathbb{D}^n$ be a holomorphic mapping. Then the following conditions are equivalent

- (a) f is a weak m -extremal,
- (b) f is an m -extremal,
- (c) f is an m -geodesic,
- (d) f_j is a non-constant Blaschke product of degree $\leq m-1$ for some $j \in \{1, \dots, n\}$.

As already mentioned, a holomorphic function $f : \mathbb{D} \rightarrow \mathbb{D}$ is a (weak) m -extremal if and only if it is a non-constant Blaschke product of degree $\leq m-1$. Thus weak m -extremality coincides with m -extremality and m -geodesity and is entirely described in all simply connected proper domains in \mathbb{C} .

Polynomial interpolation shows immediately that \mathbb{C}^n , $(\mathbb{C}_*)^k$ and $\mathbb{C}^n \times (\mathbb{C}_*)^k$ do not have weak m -extremals.

We present a description of weak m -extremals of remaining planar domains, that is domains $D \subset \mathbb{C}$ such that $\#(\mathbb{C} \setminus D) \geq 2$ and D is not biholomorphic to \mathbb{D} . These are all non-simply connected taut domains on the plane. We start with the following

Lemma 2.4. *Let $\Pi : \tilde{D} \rightarrow D$ be a holomorphic covering between domains $\tilde{D}, D \subset \mathbb{C}^n$. Assume that $\tilde{f} : \mathbb{D} \rightarrow \tilde{D}$ is an m -extremal. Then $f := \Pi \circ \tilde{f} : \mathbb{D} \rightarrow D$ is a weak m -extremal.*

Proof. Suppose that f is not a weak m -extremal. Then for any $k \geq m$ there exist $r_k > 1$ and a function $h_k \in \mathcal{O}(r_k \mathbb{D}, D)$ with $h_k(j/k) = f(j/k)$, $j = 0, \dots, m-1$. Since $\tilde{f}(0) \in \Pi^{-1}(\{h_k(0)\})$, we may lift h_k by Π to $\tilde{h}_k \in \mathcal{O}(r_k \mathbb{D}, \tilde{D})$ with the condition $\tilde{h}_k(0) = \tilde{f}(0)$. By the Montel theorem, some subsequence \tilde{h}_{l_k} is locally uniformly convergent on \mathbb{D} . Then for big k all the points $\tilde{h}_{l_k}(j/l_k)$, $\tilde{f}(j/l_k)$ ($j = 0, \dots, m-1$) drop into a neighborhood of $\tilde{f}(0)$, on which Π is biholomorphic. From

$$\Pi(\tilde{h}_{l_k}(j/l_k)) = h_{l_k}(j/l_k) = f(j/l_k) = \Pi(\tilde{f}(j/l_k))$$

we infer that $\tilde{h}_{l_k}(j/l_k) = \tilde{f}(j/l_k)$, $j = 0, \dots, m-1$, which contradicts m -extremality of \tilde{f} . \square

Proposition 2.5. *Let $D \subset \mathbb{C}$ be a non-simply connected taut domain and let $\Pi : \mathbb{D} \rightarrow D$ be a holomorphic covering. Then a holomorphic function $f : \mathbb{D} \rightarrow D$ is a weak m -extremal if and only if $f = \Pi \circ B$, where B is a non-constant Blaschke product of degree $\leq m - 1$. Moreover, f is not an m -extremal.*

Proof. Any holomorphic function $f : \mathbb{D} \rightarrow D$ can be lifted by Π , i.e. there exists $B \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ with $f = \Pi \circ B$. Assume that f is a weak m -extremal for $\lambda_1, \dots, \lambda_m$. Then B is a weak m -extremal for $\lambda_1, \dots, \lambda_m$, that is a non-constant Blaschke product of degree $\leq m - 1$.

Reversely, assume that $f = \Pi \circ B$, where B is a non-constant Blaschke product of degree $\leq m - 1$. By Lemma 2.4, the function f is a weak m -extremal.

Suppose that f is an m -extremal. We claim that Π is an m -extremal. Indeed, suppose contrary. It follows that there exist distinct points $\lambda_1, \dots, \lambda_m \in \mathbb{D}$ and a holomorphic function $h : \mathbb{D} \rightarrow D$ with $h(\lambda_j) = \Pi(\lambda_j)$, $j = 1, \dots, m$, and $h(\mathbb{D}) \subset \subset D$. Let $\mu_j \in \mathbb{D}$ be such that $\lambda_j = B(\mu_j)$. Then $h \circ B$ gives a contradiction with weak m -extremality of f for μ_1, \dots, μ_m . Since Π is of infinite (countable) multiplicity, for any $a \in D$ the set $\Pi^{-1}(\{a\}) \subset \mathbb{D}$ is infinite. Therefore, the constant function a interpolates Π for any different numbers $\lambda_1, \dots, \lambda_m \in \Pi^{-1}(\{a\})$, contradiction. \square

3. QUASI-BALANCED PSEUDOCONVEX DOMAINS

Given $k = (k_1, \dots, k_n) \in (\mathbb{N}_0^n)_*$. A domain $D \subset \mathbb{C}^n$ such that

$$\lambda \in \overline{\mathbb{D}}, z \in D \implies (\lambda^{k_1} z_1, \dots, \lambda^{k_n} z_n) \in D,$$

is called k -balanced or generally *quasi-balanced*. A $(1, \dots, 1)$ -balanced domain is *balanced*.

Lemma 3.1. *Let $D \subset \mathbb{C}^n$ be a k -balanced pseudoconvex domain and let $f \in \mathcal{O}(\mathbb{D}, D)$ (resp. $f \in \mathcal{O}(\overline{\mathbb{D}}, D)$). Assume that*

$$f = (m_\alpha^{k_1} \varphi_1, \dots, m_\alpha^{k_n} \varphi_n),$$

where $\varphi_j \in \mathcal{O}(\mathbb{D})$ (resp. $\varphi_j \in \mathcal{O}(\overline{\mathbb{D}})$), $\alpha \in \mathbb{D}$ and $\varphi := (\varphi_1, \dots, \varphi_n)$. Then either $\varphi(\mathbb{D}) \subset D$ or $\varphi(\mathbb{D}) \subset \partial D$ (resp. $\varphi \in \mathcal{O}(\overline{\mathbb{D}}, D)$).

Proof. Consider two cases.

(a) $k_1, \dots, k_n \geq 1$. Let

$$h(z) := \inf \left\{ t > 0 : \left(\frac{z_1}{t^{k_1}}, \dots, \frac{z_n}{t^{k_n}} \right) \in D \right\}, \quad z \in \mathbb{C}^n,$$

stand for the k -Minkowski function of the domain D . If $k = (1, \dots, 1)$, we have the classical Minkowski function. Then

- $D = \{z \in \mathbb{C}^n : h(z) < 1\}$,
- $h(\lambda^{k_1} z_1, \dots, \lambda^{k_n} z_n) = |\lambda| h(z)$, $z \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$,
- D is pseudoconvex if and only if $\log h \in \mathcal{PSH}(\mathbb{C}^n)$ [25] (cf. [13, Proposition 2.2.15]).

We have

$$\limsup_{\lambda \rightarrow \mathbb{T}} h(\varphi(\lambda)) = \limsup_{\lambda \rightarrow \mathbb{T}} h(f(\lambda)) \leq 1$$

(resp. $h \circ \varphi = h \circ f < 1$ on \mathbb{T}),

whence either $\varphi(\mathbb{D}) \subset D$ or $\varphi(\mathbb{D}) \subset \partial D$ (resp. $\varphi \in \mathcal{O}(\overline{\mathbb{D}}, D)$).

(b) In the opposite case assume that $k_1 = \dots = k_s = 0$, $k_{s+1}, \dots, k_n \geq 1$, where $1 \leq s \leq n - 1$. Denote $z' := (z_1, \dots, z_s)$, $z'' := (z_{s+1}, \dots, z_n)$ for $z \in \mathbb{C}^n$. Let G be the projection of D on \mathbb{C}^s . Define h by the same formula as before, but for $z \in G \times \mathbb{C}^{n-s}$. Further we proceed analogously as in [25] (cf. [13, Proposition 2.2.15]). We define the map $\Phi : G \times \mathbb{C}^{n-s} \rightarrow \mathbb{C}^n$ as $\Phi(z) := (z', z_{s+1}^{k_{s+1}}, \dots, z_n^{k_n})$

and put $\tilde{D} := \Phi^{-1}(D)$, $\tilde{h} := h \circ \Phi$. Then $\tilde{h}(z', \lambda z'') = |\lambda| \tilde{h}(z', z'')$, which means that

$$\tilde{D} = \{(z', z'') \in G \times \mathbb{C}^{n-s} : \tilde{h}(z', z'') < 1\}$$

is a pseudoconvex Hartogs domain over G with balanced fibers. For any point $z' \in G$, the function $\tilde{h}(z', \cdot)$ is the Minkowski function of the fiber

$$\tilde{D}_{z'} := \{z'' \in \mathbb{C}^{n-s} : (z', z'') \in \tilde{D}\},$$

hence G is pseudoconvex and $\log \tilde{h} \in \mathcal{PSH}(G \times \mathbb{C}^{n-s})$ [10, Proposition 4.1.14]. As $h(z) = \tilde{h}(z', \sqrt[k_{s+1}]{z_{s+1}}, \dots, \sqrt[k_n]{z_n})$ (with an arbitrary choice of the roots), we have $\log h \in \mathcal{PSH}(G \times (\mathbb{C}_*)^{n-s})$. From the removable singularities theorem it follows that $\log h \in \mathcal{PSH}(G \times \mathbb{C}^{n-s})$.

We finish the proof as in (a). \square

The following lemma will be crucial in the study of (weak) m -extremals e.g. in complex ellipsoids.

Lemma 3.2. *Let $D \subset \mathbb{C}^n$ be a k -balanced pseudoconvex domain and let $f : \mathbb{D} \rightarrow D$ be a weak m -extremal for $\lambda_1, \dots, \lambda_m$.*

- (a) *Assume that $f = (m_{\alpha}^{k_1} \varphi_1, \dots, m_{\alpha}^{k_n} \varphi_n)$, $\varphi_j \in \mathcal{O}(\mathbb{D})$, $\alpha \in \mathbb{D}$, $\varphi := (\varphi_1, \dots, \varphi_n)$. Then either $\varphi(\mathbb{D}) \subset D$ or $\varphi(\mathbb{D}) \subset \partial D$, and in the first case*
 - (i) *φ is a weak m -extremal for $\lambda_1, \dots, \lambda_m$.*
 - (ii) *if $m \geq 3$, $\lambda_m = \alpha$ and $k_1, \dots, k_n \geq 1$, then φ is a weak $(m-1)$ -extremal for $\lambda_1, \dots, \lambda_{m-1}$.*
- (b) *Suppose that $\mathbb{D} \ni \lambda_{m+1} \neq \lambda_1, \dots, \lambda_m$ and $k_1, \dots, k_n \leq 1$, $l \in \mathbb{N}$. Then the map $\psi_{(l)} := (m_{\lambda_{m+1}}^{lk_1} f_1, \dots, m_{\lambda_{m+1}}^{lk_n} f_n) : \mathbb{D} \rightarrow D$ is a weak $(m+1)$ -extremal for $\lambda_1, \dots, \lambda_{m+1}$.*

Proof. (a) Assume that $\varphi(\mathbb{D}) \subset D$.

- (i) Suppose that there exists $h \in \mathcal{O}(\overline{\mathbb{D}}, D)$ with $h(\lambda_j) = \varphi(\lambda_j)$, $j = 1, \dots, m$. Then $g := (m_{\alpha}^{k_1} h_1, \dots, m_{\alpha}^{k_n} h_n) \in \mathcal{O}(\overline{\mathbb{D}}, D)$ satisfies $g(\lambda_j) = f(\lambda_j)$, $j = 1, \dots, m$, contradiction.
- (ii) Assume that there is $h \in \mathcal{O}(\overline{\mathbb{D}}, D)$ with $h(\lambda_j) = \varphi(\lambda_j)$, $j = 1, \dots, m-1$. Then $g := (m_{\alpha}^{k_1} h_1, \dots, m_{\alpha}^{k_n} h_n) \in \mathcal{O}(\overline{\mathbb{D}}, D)$ satisfies $g(\lambda_j) = f(\lambda_j)$, $j = 1, \dots, m-1$, and $g(\alpha) = f(\alpha) = 0$, contradiction.

(b) We proceed inductively on l . For $l = 1$ assume the existence of a mapping $h \in \mathcal{O}(\overline{\mathbb{D}}, D)$ such that $h(\lambda_j) = \psi_{(1)}(\lambda_j)$, $j = 1, \dots, m+1$. The mapping $g := (h_1/m_{\lambda_{m+1}}^{k_1}, \dots, h_n/m_{\lambda_{m+1}}^{k_n}) \in \mathcal{O}(\overline{\mathbb{D}}, D)$ satisfies $g(\lambda_j) = f(\lambda_j)$, $j = 1, \dots, m+1$, contradiction.

Step $l \Rightarrow l+1$: proceed as above for $\psi_{(l)}$ and $\psi_{(l+1)}$ instead of f and $\psi_{(1)}$ respectively. \square

Assuming that f is an m -extremal, it seems that generally $\psi_{(1)}$ should not be an $(m+1)$ -extremal (P3).

Lemmas 3.2(a)(ii) and 2.1(b) give

Corollary 3.3. *Let $D \subset \mathbb{C}^n$ be a k -balanced pseudoconvex domain and let $f : \mathbb{D} \rightarrow D$ be an m -extremal. Assume that $f = (m_{\alpha}^{k_1} \varphi_1, \dots, m_{\alpha}^{k_n} \varphi_n)$, $\varphi_j \in \mathcal{O}(\mathbb{D})$, $\alpha \in \mathbb{D}$, $m \geq 3$, $k_1, \dots, k_n \geq 1$. Then either $\varphi := (\varphi_1, \dots, \varphi_n)$ is an $(m-1)$ -extremal of D or $\varphi(\mathbb{D}) \subset \partial D$.*

The question arises, whether the analogue of Corollary 3.3 holds for m -geodesics.

Remark 3.4. We shall show that it is false even in the convex case for

- (a) $m \geq 4$ and some $k \neq (1, \dots, 1)$ (Corollary 4.3).
- (b) $m \geq 4$ and $k = (1, \dots, 1)$ (Proposition 4.4).

(c) $m \geq 5$ and $k = (1, \dots, 1)$ in some complex ellipsoid (Proposition 4.5).

It would be interesting to decide what happens for $m = 4$ and $k = (1, \dots, 1)$ in (not necessarily convex) complex ellipsoids (P4).

It turns out the answer for $m = 3$ is positive.

Proposition 3.5. *Let $D \subset \mathbb{C}^n$ be a k -balanced pseudoconvex domain and let $f : \mathbb{D} \rightarrow D$ be a 3-geodesic. Assume that $f = (m_\alpha^{k_1} \varphi_1, \dots, m_\alpha^{k_n} \varphi_n)$, $\varphi_j \in \mathcal{O}(\mathbb{D})$, $\alpha \in \mathbb{D}$, $k_1, \dots, k_n \geq 1$. Then either $\varphi := (\varphi_1, \dots, \varphi_n)$ is a 2-geodesic of D or $\varphi(\mathbb{D}) \subset \partial D$.*

If f is additionally a 2-geodesic, then $\varphi(\mathbb{D}) \subset \partial D$.

Before showing it, recall

Theorem 3.6 ([7], Theorem 3). *Let $D \subset \mathbb{C}^n$ be a k -balanced pseudoconvex domain and let $f : \mathbb{D} \rightarrow D$ be a 2-geodesic. Assume that $f = (m_\alpha^{k_1} \varphi_1, \dots, m_\alpha^{k_n} \varphi_n)$, $\varphi_j \in \mathcal{O}(\mathbb{D})$, $\alpha \in \mathbb{D}$. Then either $\varphi := (\varphi_1, \dots, \varphi_n)$ is a 2-geodesic of D or $\varphi(\mathbb{D}) \subset \partial D$.*

We will proceed very similarly as in that proof.

Proof of Proposition 3.5. We can assume that $\alpha = 0$, so $f(0) = 0$. We know from Lemma 3.1 that either $\varphi \in \mathcal{O}(\mathbb{D}, D)$ or $\varphi \in \mathcal{O}(\mathbb{D}, \partial D)$. Suppose that the first case holds. Let $F \in \mathcal{O}(D, \mathbb{D})$ be such that $F \circ f$ is a Blaschke product of degree 1 or 2. One may assume that $F(0) = 0$, thus either

$$F(f(\lambda)) = \lambda, \quad \text{then denote } m := 1,$$

or

$$F(f(\lambda)) = \lambda m_\gamma(\lambda) \text{ for some } \gamma \in \mathbb{D}, \quad \text{then } m := m_\gamma.$$

Fix $z \in D$ and consider holomorphic functions defined on a neighborhood of $\overline{\mathbb{D}}$

$$g_z : \lambda \mapsto F(\lambda^{k_1} z_1, \dots, \lambda^{k_n} z_n) / \lambda, \quad m : \lambda \mapsto m(\lambda).$$

Since $|g_z(\lambda)| < 1 = |m(\lambda)|$ for $\lambda \in \mathbb{T}$, the Rouché theorem implies that the function $\mathbb{D} \ni \lambda \mapsto g_z(\lambda) - m(\lambda) \in \mathbb{C}$ has in \mathbb{D} the same number of zeros as m .

Therefore, it has no zeros if $m = 1$. This fails for $z \in \varphi(\mathbb{D})$, so the assumption $\varphi(\mathbb{D}) \subset D$ is false in that case. The ‘additionally’ claim is proved.

If $m = m_\gamma$, then the function $g_z(\lambda) - m(\lambda)$ has in \mathbb{D} exactly one root $G(z)$. Since the graph of the function $G : D \rightarrow \mathbb{D}$, equal to

$$\{(z, \lambda) \in D \times \mathbb{D} : F(\lambda^{k_1} z_1, \dots, \lambda^{k_n} z_n) = \lambda m_\gamma(\lambda)\}$$

is an analytic set, we get that G is holomorphic ([21, Chapter V, §1], cf. [5] and [11, Sekcja 5.5]). Moreover, it follows from the definition that $G(\varphi(\lambda)) = \lambda$ for $\lambda \in \mathbb{D}$, which finishes the proof. \square

We finish the section with the following property.

Lemma 3.7. *Let $D \subset \mathbb{C}^n$ be a k -balanced pseudoconvex domain and let $\varphi \in \mathcal{O}(\mathbb{D}, \partial D)$, $\alpha \in \mathbb{D}$, $k_1, \dots, k_n \leq 1$. Then $f := (m_\alpha^{k_1} \varphi_1, \dots, m_\alpha^{k_n} \varphi_n) : \mathbb{D} \rightarrow D$ is a weak 2-extremal for α and $\mu \in \mathbb{D} \setminus \{\alpha\}$.*

In particular, in the balanced case for any $a \in \partial D$ the map $\mathbb{D} \ni \lambda \mapsto \lambda a \in D$ is a weak 2-extremal for 0 and $\mu \in \mathbb{D}_$.*

Proof. One can assume that $\alpha = 0$ and $f(\lambda) = (\lambda \psi(\lambda), \tilde{\psi}(\lambda))$, $\lambda \in \mathbb{D}$, where $\psi = (\varphi_1, \dots, \varphi_s)$, $\tilde{\psi} = (\varphi_{s+1}, \dots, \varphi_n)$ for some $1 \leq s \leq n$. Suppose that $h \in \mathcal{O}(\overline{\mathbb{D}}, D)$ satisfies $h(0) = (0, \tilde{\psi}(0))$ and $h(\mu) = (\mu \psi(\mu), \tilde{\psi}(\mu))$. Then $h(\lambda) = (\lambda g(\lambda), \tilde{g}(\lambda))$, $\lambda \in \mathbb{D}$, for some map $(g, \tilde{g}) \in \mathcal{O}(\overline{\mathbb{D}}, D)$. This contradicts the equality $(g, \tilde{g})(\mu) = (\psi(\mu), \tilde{\psi}(\mu)) = \varphi(\mu) \in \partial D$. \square

4. COMPLEX ELLIPSOIDS

Let $p = (p_1, \dots, p_n) \in \mathbb{R}_{>0}^n$. The domain

$$\mathcal{E}(p) := \{z \in \mathbb{C}^n : |z_1|^{2p_1} + \dots + |z_n|^{2p_n} < 1\}$$

is said to be a *complex ellipsoid*. Write moreover

$$\mathcal{E}(\underbrace{p_0, \dots, p_0}_n) =: \mathcal{E}(p_0) \subset \mathbb{C}^n, \quad p_0 > 0,$$

for *symmetric* complex ellipsoids. The unit Euclidean ball, shortly the *ball*, is clearly

$$\mathbb{B}_n := \mathcal{E}(1) \subset \mathbb{C}^n.$$

Remark 4.1. (a) $\mathcal{E}(p)$ is k -balanced and pseudoconvex, $k \in (\mathbb{N}_0^n)_*$.

(b) If $n \geq 2$, then $\mathcal{E}(p)$ is convex if and only if $p_1, \dots, p_n \geq 1/2$.

(c) If $n \geq 2$, then $\mathcal{E}(p)$ is \mathcal{C}^2 -smooth if and only if $p_1, \dots, p_n \geq 1$.

In [18, Proposition 11] are given m -extremals being not m -geodesics for $m \geq 4$ in \mathbb{B}_n , $n \geq 2$. In Section 5 we show that it is not possible in the ball for $m = 3$. Below we have in particular 3-extremals, which are not 3-geodesics in a convex domain (Proposition 4.4 delivers other ones).

Proposition 4.2. *Let $m \geq 3$ and $0 < a < 1$. Then the map*

$$f(\lambda) := (a\lambda^{m-2}, (1-a)\lambda^{m-1}), \quad \lambda \in \mathbb{D},$$

is an m -extremal, but not an m -geodesic of $\mathcal{E}(1/2) \subset \mathbb{C}^2$.

Proof. The mapping

$$\mathbb{D} \ni \lambda \mapsto (a\lambda^{m-1}, (1-a)\lambda^{m-1}) \in \mathcal{E}(1/2)$$

is an m -geodesic (the m -left inverse $z \mapsto z_1 + z_2$), so Lemma 3.2(a)(i) says that f is an m -extremal. Suppose that there exists a holomorphic function $F : \mathcal{E}(1/2) \rightarrow \mathbb{D}$ such that $F \circ f$ is a non-constant Blaschke product of degree $\leq m-1$. We can assume that $F(0) = 0$, whence due to the Taylor expansion it follows that (with exactness up to a unimodular constant) either $F(f(\lambda)) = \lambda^{m-2}$ or $F(f(\lambda)) = \lambda^{m-2}m_\gamma(\lambda)$ for some $\gamma \in \mathbb{D}$.

In the first case we have $F(z) = z_1/a$, which is impossible.

For the second case expand $F(z) = \alpha z_1 + \beta z_2 + \delta z_1^2 + \dots$. With fixed $z \in \mathcal{E}(1/2)$, the function $g_z(\lambda) := F(\lambda z)/\lambda$, defined in a neighborhood of $\overline{\mathbb{D}}$, is smaller than 1 in modulus on \mathbb{T} . It follows that $g_z(0) = \alpha z_1 + \beta z_2 \in \mathbb{D}$ for $z \in \mathcal{E}(1/2)$. Hence $|\alpha|, |\beta| \leq 1$. From the comparison of the coefficients in the equation

$$\lambda^{m-2}m_\gamma(\lambda) = F(f(\lambda)), \quad \lambda \in \mathbb{D},$$

we have

$$\begin{aligned} -\gamma &= \alpha a, \\ 1 - |\gamma|^2 &= \begin{cases} \beta(1-a) + \delta a^2, & m = 3, \\ \beta(1-a), & m \geq 4. \end{cases} \end{aligned}$$

Consider first the possibility $m \geq 4$. We obtain

$$1 = |\alpha|^2 a^2 + \beta(1-a) \leq a^2 + 1 - a, \tag{4.1}$$

contradiction.

For $m = 3$ let the function $g : \mathbb{D} \rightarrow \overline{\mathbb{D}}$ be given by $g(z_1) := F(z_1, 0)/z_1 = \alpha + \delta z_1 + \dots$

If $|\alpha| = 1$, then g is constant, in particular $\delta = 0$. To get a contradiction use (4.1) (or note that $|\gamma| = a$, so $\beta = 1 + a$).

Otherwise g has values in \mathbb{D} . The function $h := m_\alpha \circ g : \mathbb{D} \rightarrow \mathbb{D}$ satisfies $h(0) = 0$, hence

$$1 \geq |h'(0)| = \frac{|\delta|}{1 - |\alpha|^2}, \quad |\alpha|^2 + |\delta| \leq 1.$$

This gives

$$1 = |\alpha|^2 a^2 + \delta a^2 + \beta(1 - a) \leq a^2 + 1 - a,$$

which finishes the proof. \square

Corollary 4.3 (cf. Remark 3.4(a)). *Let $m \geq 4$ and $0 < a < 1$. Then the mapping $f : \mathbb{D} \rightarrow \mathcal{E}(1/2) \subset \mathbb{C}^2$,*

$$f(\lambda) := (a\lambda^{m-1}, (1-a)\lambda^{m-1}), \quad \lambda \in \mathbb{D},$$

is an m -geodesic such that $\varphi(\lambda) := (f_1(\lambda)/\lambda^2, f_2(\lambda)/\lambda)$ is not an $(m-1)$ -geodesic.

Proposition 4.4 (cf. Remark 3.4(b)). *Let $m \geq 4$ and let numbers $a, b > 0$ be such that $4a^2 + b = 1$. Then the mapping*

$$f : \mathbb{D} \rightarrow D := \{z \in \mathbb{C}^3 : (|z_1| + |z_2|)^2 + |z_3| < 1\},$$

$$f(\lambda) := (a\lambda, a\lambda^{m-2}, b\lambda^{m-1}),$$

is an m -geodesic such that $\varphi(\lambda) := f(\lambda)/\lambda$ is not an $(m-1)$ -geodesic.

Proof. The polynomial $4z_1z_2 + z_3$ is an m -left inverse of f . Suppose that there is $F \in \mathcal{O}(D, \mathbb{D})$ such that

$$F(a, a\lambda^{m-3}, b\lambda^{m-2}) = B(\lambda), \quad \lambda \in \mathbb{D},$$

where B is a non-constant Blaschke product of degree $\leq m-2$. The function

$$G : \{(z_2, z_3) \in \mathbb{C}^2 : (a + |z_2|)^2 + |z_3| < 1\} \ni (z_2, z_3) \mapsto F(a, z_2, z_3) \in \mathbb{D}$$

satisfies

$$G(a\lambda^{m-3}, b\lambda^{m-2}) = B(\lambda), \quad \lambda \in \mathbb{D}.$$

Assume that $G(0) = 0$ and expand $G(z_2, z_3) = \alpha z_2 + \beta z_3 + \delta z_2^2 + \dots$. By considering the functions

$$g : \mathbb{D} \ni z_2 \mapsto G((1-a)z_2, 0)/z_2 \in \overline{\mathbb{D}},$$

$$\mathbb{D} \ni z_3 \mapsto G(0, (1-a^2)z_3)/z_3 \in \overline{\mathbb{D}},$$

we get that

$$|\alpha|(1-a) \leq 1, \quad |\beta|(1-a^2) \leq 1.$$

We can assume that either $B(\lambda) = \lambda^{m-3}$ or $B(\lambda) = \lambda^{m-3}m_\gamma(\lambda)$ for some $\gamma \in \mathbb{D}$.

In the first case it follows that $\alpha a = 1 \geq |\alpha|(1-a)$, i.e. $2a \geq 1$. This is impossible.

In the second one the following equations hold

$$-\gamma = \alpha a,$$

$$1 - |\gamma|^2 = \begin{cases} \beta b + \delta a^2, & m = 4, \\ \beta b, & m \geq 5. \end{cases}$$

If $m \geq 5$, note that

$$1 = |\alpha|^2 a^2 + \beta b \leq \frac{1}{(1-a)^2} a^2 + \frac{1}{(1-a^2)} (1 - 4a^2), \quad (4.2)$$

which reduces to $1 \leq 2a$, contradiction.

For $m = 4$ let us come back to the function

$$g(z_2) = \alpha(1-a) + \delta(1-a)^2 z_2 + \dots$$

If $|\alpha(1-a)| = 1$, then g is constant, in particular $\delta = 0$. Now use (4.2).

Otherwise the function $h := m_{\alpha(1-a)} \circ g : \mathbb{D} \rightarrow \mathbb{D}$ satisfies $h(0) = 0$. Hence

$$1 \geq |h'(0)| = \frac{|\delta|(1-a)^2}{1-|\alpha|^2(1-a)^2}, \quad |\alpha|^2 + |\delta| \leq \frac{1}{(1-a)^2}.$$

This gives

$$1 = |\alpha|^2 a^2 + \delta a^2 + \beta b \leq \frac{1}{(1-a)^2} a^2 + \frac{1}{(1-a^2)} (1-4a^2),$$

i.e. $1 \leq 2a$. □

A description of 2-geodesics in D (and in similar domains) may be found in [31].

Proposition 4.5 (cf. Remark 3.4(c)). *Let $m \geq 5$ and let positive numbers a, b satisfy $2a^2 + b = 1$. Then the map*

$$f : \mathbb{D} \rightarrow \mathcal{E} := \{z \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 + |z_3| < 1\},$$

$$f(\lambda) := (a\lambda, a\lambda^{m-2}, b\lambda^{m-1}),$$

is an m -geodesic such that $\varphi(\lambda) := f(\lambda)/\lambda$ is not an $(m-1)$ -geodesic.

Proof. The polynomial $2z_1 z_2 + z_3$ is an m -left inverse of f . Assume that there is a holomorphic function $F : \mathcal{E} \rightarrow \mathbb{D}$ such that

$$F(a, a\lambda^{m-3}, b\lambda^{m-2}) = B(\lambda), \quad \lambda \in \mathbb{D},$$

where B is a non-constant Blaschke product of degree at most $m-2$. Consider the function

$$G : \{(z_2, z_3) \in \mathbb{C}^2 : |z_2|^2 + |z_3| < 1\} \ni (z_2, z_3) \mapsto F(a, \sqrt{1-a^2}z_2, (1-a^2)z_3) \in \mathbb{D}.$$

Then

$$G(c\lambda^{m-3}, d\lambda^{m-2}) = B(\lambda), \quad \lambda \in \mathbb{D},$$

for some positive numbers c, d satisfying $c^2 + d = 1$, namely

$$c := \frac{a}{\sqrt{1-a^2}}, \quad d := \frac{b}{1-a^2}.$$

We may assume additionally that $G(0) = 0$. Then $B(\lambda) = \lambda^{m-3} m_\gamma(\lambda)$ for some $\gamma \in \mathbb{D}$ (with exactness up to a unimodular constant; the case $B(\lambda) = \lambda^{m-3}$ does not hold). Expanding $G(z_2, z_3) = \alpha z_2 + \beta z_3 + \dots$, we get

$$\alpha c = -\gamma,$$

$$\beta d = 1 - |\gamma|^2.$$

Therefore, $\beta(1-c^2) = 1 - |\alpha|^2 c^2$ or

$$\beta(1-c^2) + |\alpha|^2 c^2 = 1. \tag{4.3}$$

Proceeding as in the proof of Proposition 4.2, we show that $\alpha z_2 + \beta z_3 \in \mathbb{D}$ for any z_2, z_3 with $|z_2|^2 + |z_3| < 1$. In particular, $|\alpha|, |\beta| \leq 1$. It is obvious that it can not be $|\alpha| = |\beta| = 1$, whence (4.3) fails. □

By the Lempert theorem, any weak 2-extremal of a convex domain is a 2-geodesic. For all m , one-dimensional counterexamples (Proposition 2.5) are easy to generalize. Namely, let $D \subset \mathbb{C}$ be a non-simply connected taut domain and let $f : \mathbb{D} \rightarrow D$ be a weak m -extremal. Take a domain $G \subset \mathbb{C}^n$ and a map $g \in \mathcal{O}(\mathbb{D}, G)$ with $g(\mathbb{D}) \subset\subset G$. Then $(f, g) : \mathbb{D} \rightarrow D \times G$ is a weak m -extremal, but not an m -extremal (Lemma 2.2). We are not able to decide whether such a situation is possible for $m \geq 3$ in a convex domain (P5).

We present a non-convex, but topologically contractible counterexample, which follows from the following theorem.

Theorem 4.6 ([32], Theorem 4.1.1). *A complex ellipsoid $\mathcal{E}(p)$ is convex if and only if*

$$\ell_{\mathcal{E}(p)}(\lambda_1 a, \lambda_2 a) = p(\lambda_1, \lambda_2), \quad a \in \partial\mathcal{E}(p), \quad \lambda_1, \lambda_2 \in \mathbb{D}.$$

Corollary 4.7. *Let $\mathcal{E}(p)$ be non-convex. Then there exists $a \in \partial\mathcal{E}(p)$ such that for any Blaschke product B of degree $m-1$, having all zeros different, the mapping $Ba : \mathbb{D} \rightarrow \mathcal{E}(p)$ is a weak m -extremal, but not an m -extremal.*

Proof. By Lemma 3.7, for any $a \in \partial D$ the map $f_a(\lambda) := \lambda a$ is a weak 2-extremal for 0 and $\mu \in \mathbb{D}_*$, so we get weak m -extremality of Ba thanks to Lemma 3.2(b). On the other side, from Theorem 4.6 it follows that there exists $a \in \partial\mathcal{E}(p)$ such that f_a is not a 2-extremal. Therefore, if Ba were an m -extremal, making use of Corollary 3.3 we would get the opposite statement. \square

A. Edigarian [6] gave a powerful tool for studying extremal problems of type (\mathcal{P}_m) . First, the author introduced a problem (\mathcal{P}) . Let $D \subset \mathbb{C}^n$ be a bounded domain. A holomorphic mapping $f : \mathbb{D} \rightarrow D$ is called an *extremal* for (\mathcal{P}) , if $\Phi_j(f) = a_j \in \mathbb{R}$, $j = 1, \dots, N$, and there is no $h \in \mathcal{O}(\mathbb{D}, D)$ such that $\Phi_j(h) = a_j$, $j = 1, \dots, N$, and $h(\mathbb{D}) \subset\subset D$; Φ_1, \dots, Φ_N are some functionals. The mappings $g \mapsto \operatorname{Re} g(\lambda_j)$ and $g \mapsto \operatorname{Im} g(\lambda_j)$, $j = 1, \dots, m$, for some distinct $\lambda_1, \dots, \lambda_m \in \mathbb{D}$ ($N = 2m$), are model examples of such functionals. In that case it is natural, due to the Cauchy formula, to count how many λ_j 's are different from 0. This number is specified by writing (\mathcal{P}_{m-1}) or (\mathcal{P}_m) (it may be defined for other problems (\mathcal{P})). We have the following relationship with weak m -extremals.

Remark 4.8 (cf. [6], Lemma 20 and [13], Remark 11.4.4). Let $D \subset \mathbb{C}^n$ be a bounded domain. Then a holomorphic map $f : \mathbb{D} \rightarrow D$ is a weak m -extremal for m non-zero points if and only if it is an extremal for model (\mathcal{P}_m) . Otherwise, if one of m points is 0, we have equivalently an extremal for model (\mathcal{P}_{m-1}) .

A theorem of A. Edigarian delivers a necessary form of extremals $f : \mathbb{D} \rightarrow \mathcal{E}(p)$ for (\mathcal{P}_{m-1}) (for convenience we write $m-1$ instead of m and change the formulation).

Theorem 4.9 ([6], Theorem 4). *Let $f : \mathbb{D} \rightarrow \mathcal{E}(p)$ be an extremal for (\mathcal{P}_{m-1}) such that $f_j \neq 0$, $j = 1, \dots, n$. Then*

$$f_j(\lambda) = a_j \prod_{k=1}^{m-1} \left(\frac{\lambda - \alpha_{kj}}{1 - \bar{\alpha}_{kj}\lambda} \right)^{r_{kj}} \left(\frac{1 - \bar{\alpha}_{kj}\lambda}{1 - \bar{\alpha}_{k0}\lambda} \right)^{1/p_j}, \quad j = 1, \dots, n, \quad (4.4)$$

where

$$\begin{aligned} a_j &\in \mathbb{C}_*, \quad \alpha_{kj} \in \overline{\mathbb{D}}, \quad \alpha_{k0} \in \mathbb{D}, \quad r_{kj} \in \{0, 1\}, \\ \sum_{j=1}^n |a_j|^{2p_j} \prod_{k=1}^{m-1} (\lambda - \alpha_{kj})(1 - \bar{\alpha}_{kj}\lambda) &= \prod_{k=1}^{m-1} (\lambda - \alpha_{k0})(1 - \bar{\alpha}_{k0}\lambda), \quad \lambda \in \mathbb{C}, \\ f &\neq \text{const.} \end{aligned}$$

Remark 4.10. Let f be of the form (4.4).

- (a) We omit the condition $r_{kj} = 1 \implies \alpha_{kj} \in \mathbb{D}$ from the paper of A. Edigarian. It has no matter for our consideration, since for $\alpha_{kj} \in \mathbb{T}$ the function $\frac{\lambda - \alpha_{kj}}{1 - \bar{\alpha}_{kj}\lambda}$ is a unimodular constant.
- (b) Originally, there is no condition $\alpha_{k0} \in \mathbb{D}$, but $\alpha_{k0} \in \overline{\mathbb{D}}$. However, if $\alpha_{\tilde{k}0} \in \mathbb{T}$ for some \tilde{k} , then from the equality

$$\sum_{j=1}^n |a_j|^{2p_j} \prod_{k=1}^{m-1} |\lambda - \alpha_{kj}|^2 = \prod_{k=1}^{m-1} |\lambda - \alpha_{k0}|^2, \quad \lambda \in \mathbb{T},$$

we deduce that for any $j = 1, \dots, n$ there exists $k_j \in \{1, \dots, m-1\}$ such that $\alpha_{k_j j} = \alpha_{\tilde{k}_0}$. Then the corresponding factor for k_j and j in (4.4) is a unimodular constant. We redefine $\alpha_{k_j j}$ and $\alpha_{\tilde{k}_0}$ to be the same element of \mathbb{D} (or remove) and repeat the procedure if needed.

(c) The map f extends to $\overline{\mathbb{D}}$. In particular, f is proper.

Proposition 4.11 ([13], Proposition 16.2.2, [14]). *Let $\mathcal{E}(p)$ be convex and let $f : \mathbb{D} \rightarrow \mathcal{E}(p)$ be a holomorphic mapping. Then, if $f_j \neq 0$, $j = 1, \dots, n$, it follows that f is a 2-extremal (i.e. a 2-geodesic) if and only if it is of the form (4.4) with $m = 2$.*

It is not known in a general situation whether mappings given by (4.4) are even some weak l -extremals (P6). Our aim is to present solutions of particular cases.

We have new examples of convex domains, in which weak m -extremality implies m -extremality. Define the set $\mathcal{S}_n \subset [1/2, \infty)^n$ as follows:

- $(1/2, \dots, 1/2) \in \mathcal{S}_n$,
- $c \geq 1, p \in \mathcal{S}_n \implies cp \in \mathcal{S}_n$,
- $p \in \mathcal{S}_n \implies \{1, p_1, \dots, p_n\}^n \subset \mathcal{S}_n$.

The set \mathcal{S}_n may be obtained in the following way. We start with the set $\{1/2\}^n$, ‘multiply’ by $c_1 \geq 1$ to get $\bigcup \{c_1/2\}^n$, ‘add’ 1 to get $\bigcup \{1, c_1/2\}^n$, ‘multiply’ by $c_2 \geq 1$ to get $\bigcup \{c_2, c_2 c_1/2\}^n$, ‘add’ 1 and so on. Therefore

$$\begin{aligned} \mathcal{S}_n &= \bigcup_{k \in \mathbb{N}, c_1, \dots, c_k \geq 1} \{c_k, c_k c_{k-1}, \dots, c_k c_{k-1} \dots c_2, c_k c_{k-1} \dots c_1/2\}^n \\ &= \bigcup_{k \in \mathbb{N}, 1 \leq b_1, \dots, b_{k-1} \leq b_k} \{b_1, \dots, b_{k-1}, b_k/2\}^n. \end{aligned}$$

Note that

$$\begin{aligned} \mathcal{S}_n &= \bigcup_{c_1, \dots, c_n \geq 1} \{c_n, c_n c_{n-1}, \dots, c_n c_{n-1} \dots c_2, c_n c_{n-1} \dots c_1/2\}^n \\ &= \bigcup_{1 \leq b_1, \dots, b_{n-1} \leq b_n} \{b_1, \dots, b_{n-1}, b_n/2\}^n. \end{aligned}$$

We have $(p_0, \dots, p_0) \in \mathcal{S}_n$ for $p_0 \geq 1/2$ and $[1, \infty)^n \subset \mathcal{S}_n$.

Proposition 4.12. *Let $p \in \mathcal{S}_n$ and let $f : \mathbb{D} \rightarrow \mathcal{E}(p)$ be a holomorphic map. Then*

- (a) *f is a weak m -extremal if and only if it is an m -extremal.*
- (b) *if $f_j \neq 0$, $j = 1, \dots, n$, it follows that f is an m -extremal if and only if it is of the form (4.4).*

In particular, (a) and (b) hold in any symmetric convex and any \mathcal{C}^2 -smooth complex ellipsoid.

Proof. (a) It suffices to show the claim for $p = (1/2, \dots, 1/2)$ and that for any $p \in [1/2, \infty)^n$

the claim holds for $p \implies$ the claim holds for cp , with arbitrary $c > 1$,

the claim holds for $p \implies$ the claim holds for any $q \in \{1, p_1, \dots, p_n\}^n$.

(The first implication is true for $p \in \mathbb{R}_{>0}^n$, but the antecedent is always false in the non-convex case, see Corollary 4.7.)

Let $p = (1/2, \dots, 1/2)$. One can assume that f is a weak m -extremal for 0 and some other $m-1$ points and $f_j \neq 0$ for any j . Then f is of the form (4.4). Losing no generality, $a_j > 0$.

Consider the map $g : \mathbb{D} \longrightarrow \mathcal{E}(1/2) \subset \mathbb{C}^n$ given by

$$g_j(\lambda) := a_j \prod_{k=1}^{m-1} \frac{\lambda - \alpha_{kj}}{1 - \bar{\alpha}_{kj}\lambda} \left(\frac{1 - \bar{\alpha}_{kj}\lambda}{1 - \bar{\alpha}_{k0}\lambda} \right)^2$$

(g is non-constant, since $\alpha_{k0} \in \mathbb{D}$). Then

$$g_j(\lambda) = a_j \prod_{k=1}^{m-1} \frac{\lambda - \alpha_{kj}}{\lambda - \alpha_{k0}} \frac{1 - \bar{\alpha}_{kj}\lambda}{1 - \bar{\alpha}_{k0}\lambda} \prod_{k=1}^{m-1} \frac{\lambda - \alpha_{k0}}{1 - \bar{\alpha}_{k0}\lambda}.$$

Putting $F(z) := z_1 + \dots + z_n$, we have

$$F(g(\lambda)) = \prod_{k=1}^{m-1} \frac{\lambda - \alpha_{k0}}{1 - \bar{\alpha}_{k0}\lambda},$$

which shows that g is an m -geodesic. After iterating Lemma 3.2(a)(i), we get m -extremality of f .

Suppose that the claim is true for p , but not for cp . One can assume that $f : \mathbb{D} \longrightarrow \mathcal{E}(cp)$ is a weak m -extremal for 0 and some other $m-1$ points and $f_j \not\equiv 0$ for any j . Then f is of the form (4.4) with cp instead of p .

There exist different points $\lambda_1, \dots, \lambda_m \in \mathbb{D}$ and a mapping $h \in \mathcal{O}(\mathbb{D}, \mathcal{E}(cp))$ with $h(\lambda_l) = f(\lambda_l)$, $l = 1, \dots, m$, and $h(\mathbb{D}) \subset \subset \mathcal{E}(cp)$. Then

$$h_j(\lambda_l) a_j^{c-1} \prod_{k=1}^{m-1} \left(\frac{1 - \bar{\alpha}_{kj}\lambda_l}{1 - \bar{\alpha}_{k0}\lambda_l} \right)^{1/p_j - 1/(cp_j)} = a_j^c \prod_{k=1}^{m-1} \left(\frac{\lambda_l - \alpha_{kj}}{1 - \bar{\alpha}_{kj}\lambda_l} \right)^{r_{kj}} \left(\frac{1 - \bar{\alpha}_{kj}\lambda_l}{1 - \bar{\alpha}_{k0}\lambda_l} \right)^{1/p_j}.$$

Note that $g : \mathbb{D} \longrightarrow \mathbb{C}^n$ defined as

$$g_j(\lambda) := h_j(\lambda) a_j^{c-1} \prod_{k=1}^{m-1} \left(\frac{1 - \bar{\alpha}_{kj}\lambda}{1 - \bar{\alpha}_{k0}\lambda} \right)^{1/p_j - 1/(cp_j)}$$

satisfies $g(\mathbb{D}) \subset \subset \mathcal{E}(p)$. Indeed, let d be determined by the equation $1/c + 1/d = 1$, that is

$$d := \frac{c}{c-1} > 0.$$

By the Hölder inequality we have

$$\begin{aligned} \sum_{j=1}^n |g_j(\lambda)|^{2p_j} &= |h_j(\lambda)|^{2p_j} |a_j|^{2(c-1)p_j} \prod_{k=1}^{m-1} \left| \frac{1 - \bar{\alpha}_{kj}\lambda}{1 - \bar{\alpha}_{k0}\lambda} \right|^{2-2/c} \\ &\leq \left(\sum_{j=1}^n |h_j(\lambda)|^{2cp_j} \right)^{1/c} \left(\sum_{j=1}^n \left(|a_j|^{2(c-1)p_j} \prod_{k=1}^{m-1} \left| \frac{1 - \bar{\alpha}_{kj}\lambda}{1 - \bar{\alpha}_{k0}\lambda} \right|^{2(c-1)/c} \right)^d \right)^{1/d} \\ &\leq C^{1/c} \left(\sum_{j=1}^n |a_j|^{2cp_j} \prod_{k=1}^{m-1} \left| \frac{1 - \bar{\alpha}_{kj}\lambda}{1 - \bar{\alpha}_{k0}\lambda} \right|^2 \right)^{1/d} \\ &\leq C^{1/c} < 1, \end{aligned}$$

where $C := \sup_{\mathbb{D}} \sum_{j=1}^n |h_j|^{2cp_j} < 1$.

It follows that $\tilde{f} : \mathbb{D} \longrightarrow \mathcal{E}(p)$ given as

$$\tilde{f}_j(\lambda) := a_j^c \prod_{k=1}^{m-1} \left(\frac{\lambda - \alpha_{kj}}{1 - \bar{\alpha}_{kj}\lambda} \right)^{r_{kj}} \left(\frac{1 - \bar{\alpha}_{kj}\lambda}{1 - \bar{\alpha}_{k0}\lambda} \right)^{1/p_j}$$

is not a weak m -extremal for $\lambda_1, \dots, \lambda_m$, contradiction.

For the second implication we may assume that $q_1 = \dots = q_s = 1$, $q_{s+1} = p_{s+1}, \dots, q_n = p_n$, $1 \leq s \leq n$, $n \geq 2$. One can assume again that $f : \mathbb{D} \longrightarrow \mathcal{E}(q)$ is

a weak m -extremal for 0 and some other $m - 1$ points and $f_j \neq 0$ for any j . Then f is of the form (4.4) with q instead of p .

Note that $\mathbb{D} \not\subset f_j(\mathbb{D})$, $j = 1, \dots, n$. Otherwise, for some j and any $\zeta \in \mathbb{T}$ we would find a sequence $t_\nu \in \mathbb{D}$, $\nu \in \mathbb{N}$, such that $f_j(t_\nu) = (1 - 1/\nu)\zeta$. Passing to a subsequence we can assume that $t_\nu \rightarrow t \in \mathbb{D}$. Then $f_j(t) = \zeta$, so $t \in \mathbb{T}$ and $f_{j'}(t) = 0$ for $j' \neq j$. Since different ζ 's give different t 's, this implies that $f_{j'}$ has infinitely many zeros on \mathbb{T} , contradiction.

Mappings A given as follows are automorphisms of $\mathcal{E}(q)$:

$$A_j(z) := \begin{cases} \tilde{A}_j(\tilde{z}), & j = 1, \dots, s, \\ z_j \left(\frac{\sqrt{1 - |\tilde{A}^{-1}(0)|^2}}{1 - \langle \tilde{z}, \tilde{A}^{-1}(0) \rangle} \right)^{1/q_j}, & j = s + 1, \dots, n, \end{cases}$$

where $\tilde{A} \in \text{Aut}(\mathbb{B}_s)$ and $\tilde{z} := (z_1, \dots, z_s)$. Fix $t_1 \in \mathbb{D} \setminus f_1(\mathbb{D})$ and consider

$$\tilde{A} := \chi_{(t_1, 0, \dots, 0)} : \tilde{z} \mapsto \left(\frac{z_1 - t_1}{1 - \bar{t}_1 z_1}, z_2 \frac{\sqrt{1 - |t_1|^2}}{1 - \bar{t}_1 z_1}, \dots, z_s \frac{\sqrt{1 - |t_1|^2}}{1 - \bar{t}_1 z_1} \right)$$

(for the general definition of χ_w consult Section 5). Then

- $A \circ f$ is a weak m -extremal for 0 and some other $m - 1$ points,
- $(A \circ f)_1$ has no zeros in \mathbb{D} ,
- $(A \circ f)_j$, $j \neq 1$, has in \mathbb{D} the same zeros as f_j ,
- $(A \circ f)_j \neq 0$ for any j .

Now take $\chi_{(0, t_2, \dots, 0)}$, where $t_2 \in \mathbb{D} \setminus (A \circ f)_2(\mathbb{D})$, and proceed similarly with $A \circ f$ instead of f . After the s -th step we get a map \tilde{f} such that $\tilde{f}_j \neq 0$ in \mathbb{D} , $j = 1, \dots, s$. It is of the form (4.4) with some \tilde{a}_j , $\tilde{\alpha}_{kj}$, \tilde{r}_{kj} , but additionally $\tilde{\alpha}_{kj} \in \mathbb{D}$ and $\tilde{r}_{kj} = 0$ for $k = 1, \dots, m - 1$, $j = 1, \dots, s$.

Suppose that there exist different points $\lambda_1, \dots, \lambda_m \in \mathbb{D}$ and a mapping $h \in \mathcal{O}(\mathbb{D}, \mathcal{E}(q))$ with $h(\lambda_l) = \tilde{f}(\lambda_l)$ for any l and $h(\mathbb{D}) \subset \subset \mathcal{E}(q)$. For $t \in \mathbb{C}$ the map $\tilde{h} := th + (1 - t)\tilde{f}$ satisfies $\tilde{h}(\lambda_l) = \tilde{f}(\lambda_l)$, $l = 1, \dots, m$. However, for $t \in (0, 1)$ close to 0 the coordinate functions \tilde{h}_j , $j = 1, \dots, s$, do not vanish in \mathbb{D} .

Define $g : \mathbb{D} \rightarrow \mathbb{C}^n$ by $g_j := \tilde{h}_j^{q_j/p_j}$. The Jensen inequality implies that

$$\begin{aligned} \sum_{j=1}^n |g_j|^{2p_j} &= \sum_{j=1}^n |th_j + (1 - t)\tilde{f}_j|^{2q_j} \\ &\leq t \sum_{j=1}^n |h_j|^{2q_j} + (1 - t) \sum_{j=1}^n |\tilde{f}_j|^{2q_j} < tC + 1 - t < 1 \text{ on } \mathbb{D}, \end{aligned}$$

where $C := \sup_{\mathbb{D}} \sum_{j=1}^n |h_j|^{2q_j} < 1$. Thus $g(\mathbb{D}) \subset \subset \mathcal{E}(p)$. Further we have

$$g_j(\lambda_l) = \zeta_j \tilde{a}_j^{q_j/p_j} \prod_{k=1}^{m-1} \left(\frac{\lambda_l - \tilde{\alpha}_{kj}}{1 - \bar{\alpha}_{kj} \lambda_l} \right)^{(q_j/p_j)\tilde{r}_{kj}} \left(\frac{1 - \bar{\alpha}_{kj} \lambda_l}{1 - \bar{\alpha}_{k0} \lambda_l} \right)^{1/p_j}$$

for some $\zeta_j \in \mathbb{T}$. This shows that the map $\tilde{g} : \mathbb{D} \rightarrow \mathcal{E}(p)$,

$$\tilde{g}_j(\lambda) := \tilde{a}_j^{q_j/p_j} \prod_{k=1}^{m-1} \left(\frac{\lambda - \tilde{\alpha}_{kj}}{1 - \bar{\alpha}_{kj} \lambda} \right)^{(q_j/p_j)\tilde{r}_{kj}} \left(\frac{1 - \bar{\alpha}_{kj} \lambda}{1 - \bar{\alpha}_{k0} \lambda} \right)^{1/p_j}$$

is not a weak m -extremal for $\lambda_1, \dots, \lambda_m$, contradiction.

(b) Corollary from the proof of (a). □

We suppose that weak m -extremality coincides with m -extremality in any convex complex ellipsoid (P7).

In a more general case we get higher extremality.

Proposition 4.13. *Let $f : \mathbb{D} \rightarrow \mathcal{E}(p)$ be given by (4.4). Assume that*

- (a) $p_1, \dots, p_n \geq 1/2$,
- (b) $q \in \mathcal{S}_n$,
- (c) $\alpha_{kj} \in \mathbb{D}$, $r_{kj} = 0$ for $k = 1, \dots, m-1$ and $j \in J$, where $J := \{j : p_j/q_j \notin \mathbb{N}\}$,
- (d) $s_j := \#\{k : r_{kj} = 1\}$,
- (e) $\tilde{m} := m + \sum_{j \notin J} (p_j/q_j - 1)s_j$.

Then f is an \tilde{m} -extremal.

In particular, f is an $(m + (m-1)(p_1/q_1 + \dots + p_n/q_n - n))$ -extremal, provided that $q \in \mathcal{S}_n$ and $p_j/q_j \in \mathbb{N}$, $j = 1, \dots, n$.

Proof. Suppose contrary. Note that $f_j \neq 0$ in $\overline{\mathbb{D}}$, $j \in J$. Proceeding like in the proof of Proposition 4.12(a) (now functions $\tilde{h}_j^{p_j/q_j}$ are well-defined), we obtain that $\tilde{g} : \mathbb{D} \rightarrow \mathcal{E}(q)$ defined by

$$\tilde{g}_j(\lambda) := a_j^{p_j/q_j} \prod_{k=1}^{m-1} \left(\frac{\lambda - \alpha_{kj}}{1 - \overline{\alpha}_{kj}\lambda} \right)^{(p_j/q_j)r_{kj}} \left(\frac{1 - \overline{\alpha}_{kj}\lambda}{1 - \overline{\alpha}_{k0}\lambda} \right)^{1/q_j}$$

is not an \tilde{m} -extremal.

On the other side, by Proposition 4.12(b), the mapping $g : \mathbb{D} \rightarrow \mathcal{E}(q)$,

$$g_j(\lambda) := a_j^{p_j/q_j} \prod_{k=1}^{m-1} \left(\frac{\lambda - \alpha_{kj}}{1 - \overline{\alpha}_{kj}\lambda} \right)^{r_{kj}} \left(\frac{1 - \overline{\alpha}_{kj}\lambda}{1 - \overline{\alpha}_{k0}\lambda} \right)^{1/q_j},$$

is an m -extremal. We use $\sum_{j \notin J} (p_j/q_j - 1)s_j$ times Lemma 3.2(b) and Proposition 4.12(a) to get \tilde{m} -extremality of \tilde{g} , contradiction. \square

Remark 4.14. Note the fact following from the proof of Proposition 4.13. Suppose that $p, q \in \mathbb{R}_{>0}^n$ are such that $p_j/q_{\sigma(j)} \in \mathbb{N}$, $j = 1, \dots, n$, for some permutation σ of $\{1, \dots, n\}$ (it is equivalent to the existence of a proper holomorphic map between $\mathcal{E}(p)$ and $\mathcal{E}(q)$). Assume that any map of the form (4.4) in $\mathcal{E}(q)$ is some (weak) t -extremal. Then any map given by (4.4) in $\mathcal{E}(p)$ is some (weak) s -extremal. However, this procedure delivers the same p as described in Proposition 4.13.

Proposition 4.15. *Let $f : \mathbb{D} \rightarrow \mathcal{E}(p)$ be of the form (4.4). Assume that*

- (a) $p_1, \dots, p_n \geq 1/2$,
- (b) $q \in \mathcal{S}_n$,
- (c) $\alpha_{kj} \in \mathbb{D}$ for $k = 1, \dots, m-1$ and $j \in J$, where $J := \{j : p_j/q_j \notin \mathbb{N}\}$,
- (d) $S := \{(k, j) : r_{kj} = 1\}$,
- (e) α_{kj} , $(k, j) \in S$, are distinct,
- (f) $s := \#S \geq m$.

Then f is a weak s -extremal for α_{kj} , $(k, j) \in S$.

Proof. Suppose that there exists a holomorphic mapping $h : \mathbb{D} \rightarrow \mathcal{E}(p)$ with $h(\alpha_{kj}) = f(\alpha_{kj})$, $(k, j) \in S$, and $h(\mathbb{D}) \subset \subset \mathcal{E}(p)$. In particular, $h_j(\alpha_{kj}) = 0$, $(k, j) \in S$. Consider the maps $g : \mathbb{D} \rightarrow \mathcal{E}(p)$ and $\tilde{f} : \mathbb{D} \rightarrow \overline{\mathcal{E}(p)}$ given as

$$g_j(\lambda) := \frac{h_j(\lambda)}{\prod_{k=1}^{m-1} \left(\frac{\lambda - \alpha_{kj}}{1 - \overline{\alpha}_{kj}\lambda} \right)^{r_{kj}}},$$

$$\tilde{f}_j(\lambda) := a_j \prod_{k=1}^{m-1} \left(\frac{1 - \overline{\alpha}_{kj}\lambda}{1 - \overline{\alpha}_{k0}\lambda} \right)^{1/p_j}.$$

We have $g(\alpha_{kj}) = \tilde{f}(\alpha_{kj})$, $(k, j) \in S$, and $g(\mathbb{D}) \subset \subset \mathcal{E}(p)$. It follows that $\tilde{f}(\mathbb{D}) \subset \mathcal{E}(p)$, as otherwise \tilde{f} would be a constant lying in the boundary of $\mathcal{E}(p)$ (it would

also contradict the condition $s \geq m$). Hence \tilde{f} is not a weak s -extremal for α_{kj} , $(k, j) \in S$. However, by Proposition 4.13, the mapping \tilde{f} is an m -extremal. This is impossible, since $s \geq m$. \square

In the sequel (see also Proposition 5.9) occur non-constant mappings of the form $(a_1 B_1, \dots, a_n B_n)$, where $a \in \partial \mathcal{E}(p)$ and B_1, \dots, B_n are finite Blaschke products. We think that any m -extremal of the ball is equivalent with some of these maps (P9), which are suspected to be some k -geodesics (P8). This would give a positive answer for (P10).

Remark 4.16. Some m -extremality of maps $(a_1 B_1, \dots, a_n B_n)$ in convex complex ellipsoids follows from 2-geodesity of the mapping $\lambda \mapsto \lambda a$ and Lemma 3.2(a)(i).

Proposition 4.17. *Let $a \in \partial \mathcal{E}(p)$ be such that*

$$(p_j |a_j|^{2p_j})_{j=1}^n = c(m_1, \dots, m_n), \quad c > 0, \quad m_j \in \mathbb{N}.$$

Assume that B_1, \dots, B_n are finite Blaschke products, not all constant. Then the map $(a_1 B_1, \dots, a_n B_n) : \mathbb{D} \rightarrow \mathcal{E}(p)$ is some m -geodesic.

Proof. Consider the logarithmic image of $\mathcal{E}(p)$, that is the convex domain

$$\begin{aligned} \Omega &:= \{x \in \mathbb{R}^n : (e^{x_1}, \dots, e^{x_n}) \in \mathcal{E}(p)\} \\ &= \left\{ x \in \mathbb{R}^n : \sum_{j=1}^n e^{2p_j x_j} < 1 \right\}. \end{aligned}$$

The affine tangent space at $b := (\log |a_1|, \dots, \log |a_n|) \in \partial \Omega$ is

$$\left\{ x \in \mathbb{R}^n : \sum_{j=1}^n p_j e^{2p_j b_j} (x_j - b_j) = 0 \right\} = \left\{ x \in \mathbb{R}^n : \sum_{j=1}^n c m_j (x_j - b_j) = 0 \right\},$$

whence

$$\Omega \subset \left\{ x \in \mathbb{R}^n : \sum_{j=1}^n m_j (x_j - b_j) < 0 \right\}.$$

This implies

$$\begin{aligned} \mathcal{E}(p) &\subset \left\{ z \in \mathbb{C}^n : \sum_{j=1}^n m_j \log |z_j| < \sum_{j=1}^n m_j b_j \right\} \\ &= \left\{ z \in \mathbb{C}^n : \prod_{j=1}^n |z_j|^{m_j} < \prod_{j=1}^n |a_j|^{m_j} \right\}, \end{aligned}$$

so the polynomial

$$F(z) := \prod_{j=1}^n \left(\frac{z_j}{a_j} \right)^{m_j}$$

is an m -left inverse we are looking for. \square

Proposition 4.18. *Let $a \in \partial \mathcal{E}(p)$ and let m be the least common multiplicity of numbers $m_1, \dots, m_n \in \mathbb{N}$. Assume that $2p_j m_j \geq m$, $j = 1, \dots, n$. Then the map $\mathbb{D} \ni \lambda \mapsto (a_1 \lambda^{m_1}, \dots, a_n \lambda^{m_n}) \in \mathcal{E}(p)$ is an $(m+1)$ -geodesic.*

Proof. One may assume that $a_j \in (0, 1)$. Define the domain

$$\begin{aligned}\Omega &:= \{x \in \mathbb{R}^n : (x_1^{m_1/m}, \dots, x_n^{m_n/m}) \in \mathcal{E}(p)\} \\ &= \left\{ x \in \mathbb{R}^n : \sum_{j=1}^n x_j^{\frac{2p_j m_j}{m}} < 1 \right\},\end{aligned}$$

which is convex. The affine tangent space at $b := (a_1^{m/m_1}, \dots, a_n^{m/m_n}) \in \partial\Omega$ is

$$\left\{ x \in \mathbb{R}^n : \sum_{j=1}^n p_j m_j b_j^{\frac{2p_j m_j}{m}-1} (x_j - b_j) = 0 \right\},$$

whence

$$\Omega \subset \left\{ x \in \mathbb{R}^n : \sum_{j=1}^n p_j m_j b_j^{\frac{2p_j m_j}{m}-1} (x_j - b_j) < 0 \right\}.$$

It follows that

$$\mathcal{E}(p) \subset \left\{ z \in \mathbb{C}^n : \sum_{j=1}^n p_j m_j b_j^{\frac{2p_j m_j}{m}-1} |z_j|^{\frac{m}{m_j}} < \sum_{j=1}^n p_j m_j b_j^{\frac{2p_j m_j}{m}} \right\},$$

so the polynomial

$$F(z) := \frac{\sum_{j=1}^n p_j m_j b_j^{\frac{2p_j m_j}{m}-1} z_j^{\frac{m}{m_j}}}{\sum_{j=1}^n p_j m_j b_j^{\frac{2p_j m_j}{m}}}$$

is an $(m+1)$ -left inverse. \square

5. THE EUCLIDEAN BALL

We say that holomorphic mappings $f, g : \mathbb{D} \rightarrow \mathbb{B}_n$ are *equivalent* if there exists $A \in \text{Aut}(\mathbb{B}_n)$ such that $f = A \circ g$.

Recall that the automorphism group of the ball consists of the mappings $U \circ \chi_w$ (equivalently, of the mappings $\chi_w \circ U$), where $U : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is unitary and $\chi_w : \mathbb{B}_n \rightarrow \mathbb{B}_n$ defined as $\chi_0 := \text{id}_{\mathbb{B}_n}$ and

$$\chi_w(z) := \frac{1}{|w|^2} \frac{\sqrt{1-|w|^2}(|w|^2 z - \langle z, w \rangle w) - |w|^2 w + \langle z, w \rangle w}{1 - \langle z, w \rangle}, \quad w \in \mathbb{B}_{n*}.$$

Remark 5.1. Any 2-extremal $f : \mathbb{D} \rightarrow \mathbb{B}_n$ is equivalent to $\lambda \mapsto (\lambda, 0, \dots, 0)$.

Remark 5.2 ([18]). (a) Any 3-extremal $f : \mathbb{D} \rightarrow \mathbb{B}_n$, $n \geq 2$, is equivalent with some map

$$g : \mathbb{D} \ni \lambda \mapsto (a\lambda, \sqrt{1-a^2}\lambda m_\alpha(\lambda), 0, \dots, 0) \in \mathbb{B}_n, \quad (5.1)$$

where $0 \leq a \leq 1$ and $\alpha \in \mathbb{D}$ (take $A \in \text{Aut}(\mathbb{B}_n)$ such that $A(f(0)) = 0$, divide by λ to get either a 2-extremal or a constant from the boundary, unitarily transform in such a way that some two points of this 2-extremal have the same first coordinate and use the form of 2-extremals).

- (b) Any map of the form (5.1) is a 3-extremal.
- (c) A mapping given by (5.1) is a 2-extremal if and only if $a = 1$.
- (d) Any 3-extremal is equivalent with exactly one map of the form (5.1).
- (e) For $\alpha = 0$ the map given by (5.1) is a 3-geodesic, since it has the 3-left inverse

$$F(z) := \frac{1}{2-a^2} z_1^2 + \frac{2\sqrt{1-a^2}}{2-a^2} z_2.$$

By the Schur's algorithm we have the following characterization.

Remark 5.3 ([18]). Let $f : \mathbb{D} \rightarrow \mathbb{B}_n$ be a holomorphic mapping. Then f is an m -extremal if and only if

$$f(\lambda) = A_1(\lambda A_2(\lambda \dots A_l(\lambda a) \dots)), \quad \lambda \in \mathbb{D},$$

for some $A_1, \dots, A_l \in \text{Aut}(\mathbb{B}_n)$, $1 \leq l \leq m-1$, $a \in \partial \mathbb{B}_n$.

In particular, any m -extremal of \mathbb{B}_n extends holomorphically to a neighborhood of $\overline{\mathbb{D}}$.

Note also that

Remark 5.4 (cf. the proof of Proposition 4.12(a)). Any m -extremal of \mathbb{B}_n , $n \geq 2$, is equivalent with a map, whose coordinates have no zeros in a neighborhood of $\overline{\mathbb{D}}$.

Recall less obvious facts.

Proposition 5.5 ([18], Proposition 8). *Any weak m -extremal of \mathbb{B}_n is an m -extremal.*

Proposition 5.6 ([18], Proposition 11). *Let $m \geq 4$ and $0 < a < 1$. Then the mapping*

$$f(\lambda) := (a\lambda^{m-2}, \sqrt{1-a^2}\lambda^{m-1}), \quad \lambda \in \mathbb{D},$$

is an m -extremal, but not an m -geodesic of \mathbb{B}_2 .

Remark 5.7. The fundamental Poincaré theorem states that \mathbb{B}_n and \mathbb{D}^n are not biholomorphic if $n \geq 2$. Note that a new proof of this fact follows from Remark 2.3 and Proposition 5.6.

The main result of the section is

Theorem 5.8. *Any 3-extremal of \mathbb{B}_n is a 3-geodesic.*

Proof. It suffices to prove the claim for $n = 2$. Consider 3-geodesics of the form

$$f(\lambda) := (am_c(\lambda), bm_c(\lambda)^2), \quad \lambda \in \mathbb{D},$$

where $a, b \in (0, 1)$, $a^2 + b^2 = 1$ and $c \in \mathbb{D}_*$. Any such mapping is equivalent to $g(\lambda) := (\alpha\lambda, \beta\lambda m_\gamma(\lambda))$ for some $\alpha, \beta \in [0, 1]$, $\alpha^2 + \beta^2 = 1$ and $\gamma \in \mathbb{D}$, i.e. there are a unitary map U and a point $w \in \mathbb{B}_2$ such that

$$\chi_w(am_c(\lambda), bm_c(\lambda)^2) = U(\alpha\lambda, \beta\lambda m_\gamma(\lambda)). \quad (5.2)$$

We will find formulas for β and γ depending of b and c . Then we shall prove that (β, γ) runs over the whole set $(0, 1) \times \mathbb{D}_*$ as (b, c) runs over it. This will let us ‘invert’ g , since we are able to do it with f .

Taking $\lambda := 0$ in (5.2) we get $w = (-ac, bc^2)$. Note that $\beta \neq 0$, since otherwise $\lambda := c$ gives $\chi_w(0) = U(c, 0)$; hence $|w|^2 = |c|^2$, i.e. $a^2 + b^2|c|^2 = 1$, contradiction.

By the formula for χ_w we have

$$\begin{aligned} p_0 + p_1 m_c(\lambda) + p_2 m_c(\lambda)^2 \\ = (1 + a^2 \bar{c} m_c(\lambda) - b^2 \bar{c}^2 m_c(\lambda)^2) \lambda (q_1 \alpha + q_2 \beta m_\gamma(\lambda), q_3 \alpha + q_4 \beta m_\gamma(\lambda)) \end{aligned} \quad (5.3)$$

for some $p_j \in \mathbb{C}^2$, $q_j \in \mathbb{C}$ with $q_2 \neq 0$ or $q_4 \neq 0$. Therefore,

$$1 + a^2 \bar{c} m_c(1/\bar{\gamma}) - b^2 \bar{c}^2 m_c(1/\bar{\gamma})^2 = 0, \quad (5.4)$$

unless $\gamma = 0$ or $\gamma = c$.

Suppose that $\gamma = 0$ and $q_2 \neq 0$. Then

$$\begin{aligned} p_{01} + p_{11}\lambda + p_{21}\lambda^2 &= (1 + (1 - b^2)\bar{c}\lambda - b^2\bar{c}^2\lambda^2)m_{-c}(\lambda)(q_1\alpha + q_2\beta m_{-c}(\lambda)) \\ &= (1 - b^2\bar{c}\lambda)(c + \lambda)(q_1\alpha + q_2\beta m_{-c}(\lambda)). \end{aligned}$$

Since the numbers

$$\frac{1}{b^2\bar{c}}, \quad -c, \quad -\frac{1}{\bar{c}}$$

are different, we infer that the right side has a singularity, contradiction.

The case $\gamma = c$ is also impossible, as otherwise the rank of the singularity $1/\bar{c}$ on the right side of (5.3) would equal 3.

The equation (5.4) is equivalent to

$$(1 - b^2\bar{c}m_c(1/\bar{\gamma}))(1 + \bar{c}m_c(1/\bar{\gamma})) = 0,$$

that is $m_c(1/\bar{\gamma}) = 1/(b^2\bar{c})$, i.e.

$$\gamma = c \frac{1 + b^2}{1 + b^2|c|^2} = m_{-c}(b^2c).$$

Moreover, there exist a unitary map \tilde{U} and a point $\tilde{w} \in \mathbb{B}_2$ satisfying

$$\tilde{U}(am_c(\lambda), bm_c(\lambda)^2) = \chi_{\tilde{w}}(\alpha\lambda, \beta\lambda m_\gamma(\lambda)), \quad \lambda \in \mathbb{D},$$

whence $0 = \chi_{\tilde{w}}(\alpha c, \beta c m_\gamma(c))$ and $\tilde{U}(-ac, bc^2) = \chi_{\tilde{w}}(0)$. This implies

$$a^2|c|^2 + b^2|c|^4 = \alpha^2|c|^2 + \beta^2|c|^2|m_\gamma(c)|^2,$$

equivalently (using $|m_\gamma(c)| = |m_c(\gamma)| = b^2|c|$)

$$1 - b^2 + b^2|c|^2 = \alpha^2 + (1 - \alpha^2)b^4|c|^2.$$

Therefore,

$$\alpha^2 = \frac{(1 - b^2)(1 + b^2|c|^2)}{1 - b^4|c|^2}, \quad \beta^2 = \frac{b^2 - b^2|c|^2}{1 - b^4|c|^2} = -m_{b^2}(b^2|c|^2).$$

To finish the proof, it suffices to show that the mapping

$$h : (0, 1) \times \mathbb{D}_* \ni (b, c) \mapsto (-m_{b^2}(b^2|c|^2), m_{-c}(b^2c)) \in (0, 1) \times \mathbb{D}_*$$

is surjective. It is equivalent to the surjectivity of

$$(0, 1)^2 \ni (b, c) \mapsto h(b, c) \in (0, 1)^2.$$

Fix $(p, q) \in (0, 1)^2$. Putting

$$F(\lambda) := m_q(\lambda) - \lambda m_p(\lambda m_q(\lambda)), \quad \lambda \in \mathbb{D},$$

we see that $F(-1, 1) \subset \mathbb{R}$, $F(0) = -q < 0$ and $F(q) = pq > 0$. Thus there exists $c \in (0, q)$ such that $F(c) = 0$. Note that $-c < m_q(c) < 0$. Let $b \in (0, 1)$ satisfy $-b^2 = m_q(c)/c$. Then $m_c(q) = b^2c$, i.e. $q = m_{-c}(b^2c)$. Moreover,

$$m_{-b^2}(-p) = -m_{-p}(-b^2) = -m_{-p}\left(\frac{m_q(c)}{c}\right) = -cm_q(c) = b^2c^2,$$

so $p = -m_{b^2}(b^2c^2)$. □

In Propositions 4.17 and 4.18 some m -geodesity of mappings (a_1B_1, \dots, a_nB_n) was investigated. We add one more positive result.

Proposition 5.9. *Let $m \geq 3$, $0 < b \leq \frac{1}{m-1}$ and $a := \sqrt{1 - b^2}$. Then the mapping $f(\lambda) := (a\lambda, b\lambda^m)$, $\lambda \in \mathbb{D}$, is an $(m + 1)$ -geodesic of \mathbb{B}_2 .*

Proof. Consider the more general situation $f(\lambda) = (a\lambda^k, b\lambda^m)$, $k \geq 1$, $m \geq 3$, and use the Lagrange multipliers to the functions of real variables $F(x, y) := cx^m + dy^k$ and $G(x, y) := x^2 + y^2 - 1$ ($c, d > 0$ specified later). We wish F had a global (weak)

maximum equal to 1 on the set $\{G = 0\}$ at the point (a, b) . Denote $H := F - tG$, where $t \in \mathbb{R}$ is fixed. From the necessary condition for a local extremum we have

$$\begin{aligned} 0 &= \frac{\partial H}{\partial x}(x, y) = mcx^{m-1} - 2tx, \\ 0 &= \frac{\partial H}{\partial y}(x, y) = kdy^{k-1} - 2ty, \\ 1 &= x^2 + y^2. \end{aligned}$$

Excluding for a moment the cases $(1, 0)$ and $(0, 1)$, we find that (remembering that $1 = ca^m + db^k$)

$$c = \frac{k}{(ka^2 + mb^2)a^{m-2}}, \quad d = \frac{m}{(ka^2 + mb^2)b^{k-2}}$$

(formally, we define c, d by these formulas). The tangent space at (a, b) is $\mathbb{R}(b, -a)$, so (a, b) is a local maximum if

$$\begin{aligned} 0 &> \frac{\partial^2 H}{\partial x^2}(a, b)b^2 + \frac{\partial^2 H}{\partial y^2}(a, b)a^2 \\ &= (m(m-1)ca^{m-2} - 2t)b^2 + (k(k-1)db^{k-2} - 2t)a^2 \\ &= 2t(m-2)b^2 + 2t(k-2)a^2. \end{aligned} \tag{5.5}$$

Since $t > 0$, we see why only $k = 1$ may work; in what follows we assume that $k = 1$. In that situation (5.5) is equivalent to $b^2 < \frac{1}{m-1}$, which is true.

It remains to check that $F(x, y) \leq 1$ for any x, y satisfying the necessary condition. First, we will show that $F(1, 0), F(0, 1) \leq 1$, that is $c, d \leq 1$. It occurs that $d \leq 1$ is equivalent to $b \leq \frac{1}{m-1}$. For the condition $c \leq 1$ we need that

$$1 \leq (a^2 + m(1 - a^2))a^{m-2} = ma^{m-2} - (m-1)a^m,$$

so consider the function $g(s) := ms^{m-2} - (m-1)s^m$. It decreases on the interval $\left[\sqrt{1 - \frac{1}{m-1}}, 1\right] \ni a$, so $g(a) > g(1) = 1$.

Now let $x, y \neq 0$ satisfy the necessary condition. Then $mcx^{m-2} = 2t = d/y$, that is

$$yx^{m-2} = \frac{d}{mc} = ba^{m-2}.$$

Define $h(s) := s\sqrt{1 - s^{2m-2}}$. Then $h(y) = h(b)$ and h increases on the interval $\left[0, \sqrt{\frac{1}{m-1}}\right] \ni b$, so $y \geq b$. Our aim is to show that $cx^m + dy \leq 1$, that is

$$\begin{aligned} \frac{x^m}{a^{m-2}} + mby &\leq a^2 + mb^2, \\ \frac{x^m b}{yx^{m-2}} + mby &\leq 1 + (m-1)b^2, \\ b(1 - y^2) + mby^2 &\leq y + (m-1)b^2y, \\ 0 &\leq ((m-1)by - 1)(b - y). \end{aligned}$$

The last inequality holds, since $(m-1)by - 1 \leq y - 1 < 0$. □

The case $\frac{1}{m-1} < b < 1$ remains unsolved (P11).

6. BOUNDARY PROPERTIES

In this section we discuss (almost) properness of weak m -extremals. Thanks to almost properness we conclude their uniqueness in bounded strictly convex domains.

Let $D \subset \mathbb{C}^n$ be a bounded domain and $f : \mathbb{D} \rightarrow D$ a holomorphic mapping. We say that f is *almost proper* if $f^*(\zeta) \in \partial D$ for almost all $\zeta \in \mathbb{T}$ with respect to the

Lebesgue measure on \mathbb{T} . As usual, $f^*(\zeta) := \lim_{r \rightarrow 1^-} f(r\zeta)$ is the non-tangential boundary value of f at ζ , which exists for almost all $\zeta \in \mathbb{T}$, see [16].

A domain $D \subset \mathbb{C}^n$ is called *weakly Runge* if it is bounded and there exists a domain $G \supset \overline{D}$ such that for any bounded holomorphic map $f : \mathbb{D} \rightarrow G$ with $f^*(\mathbb{T}) \subset\subset D$ we have $f(\mathbb{D}) \subset\subset D$.

Remark 6.1 ([8], Remark 2). (a) A bounded Runge domain is weakly Runge.
(b) Let $G \subset \mathbb{C}^n$ be a domain and let u be a plurisubharmonic function in G . Assume that

$$D := \{z \in G : u(z) < 0\} \subset\subset G.$$

Then any component of D is a weakly Runge domain.

Proposition 6.2 (cf. [8], Theorem 1). *Let $D \subset \mathbb{C}^n$ be a weakly Runge domain and let $f : \mathbb{D} \rightarrow D$ be a weak m -extremal such that for some $\gamma > 0$ we have*

$$\text{dist}(f(\lambda), \partial D) \geq \gamma(1 - |\lambda|), \quad \lambda \in \mathbb{D}.$$

Then for any $\alpha > 0$ and $\beta < 1$ the set

$$Q(\alpha, \beta) := \{\zeta \in \mathbb{T} : \text{dist}(f(t\zeta), \partial D) \geq \alpha(1 - t)^\beta \text{ for any } t \in (0, 1)\}$$

has Lebesgue measure zero on \mathbb{T} . In particular, f is almost proper.

Proof. This is a slight modification of the proof of [8, Theorem 1]. For the Reader's convenience, we present the whole proof (the first and the last part are mostly copied).

Note that for $\beta_1 < \beta_2$ we have $Q(\alpha, \beta_1) \subset Q(\alpha, \beta_2)$. Without loss of generality one may assume that for some $\alpha > 0$ and $\beta \in (0, 1)$ the set $P := Q(\alpha, \beta)$ has positive measure. We can assume that

$$0 < \frac{1}{2\pi} \int_{\{\theta \in (0, 2\pi) : e^{i\theta} \in P\}} d\theta < 1$$

(otherwise we take as P any subset of $Q(\alpha, \beta)$ of positive measure). We put

$$\varphi(\lambda) := \frac{1}{2\pi} \int_{\{\theta \in (0, 2\pi) : e^{i\theta} \in P\}} \frac{e^{i\theta} + \lambda}{e^{i\theta} - \lambda} d\theta, \quad \lambda \in \mathbb{D},$$

and check that $\text{Re } \varphi(\lambda) > 0$ and $\text{Re}(1 - \varphi(\lambda)) > 0$. In particular, φ^* exists almost everywhere [16, Chapter III, Section C].

Losing no generality assume that f is a weak m -extremal for $\lambda_1, \dots, \lambda_{m-1}, 0$. For $t \in (0, 1)$ define

$$h_t(\lambda) := f(t\lambda) + \sum_{j=1}^{m-1} \left(e^{\gamma_t(\varphi(\lambda) - \varphi(\lambda_j))} \frac{\lambda}{\lambda_j} \prod_{k \neq j} \frac{\lambda - \lambda_k}{\lambda_j - \lambda_k} \right) (f(\lambda_j) - f(t\lambda_j)), \quad \lambda \in \mathbb{D},$$

with $\gamma_t \in \mathbb{R}$ specified later. Then $h_t(\lambda_l) = f(\lambda_l)$ for any l and $h_t(0) = f(0)$. Our aim is to show that for all $t \in (0, 1)$ sufficiently close to 1 there exists γ_t such that $h_t(\mathbb{D}) \subset\subset D$. First, we shall prove that $h_t^*(\mathbb{T}) \subset\subset D$.

It is sufficient to have for t close to 1

$$\sum_{j=1}^{m-1} e^{\gamma_t(\text{Re } \varphi^*(\zeta) - \text{Re } \varphi(\lambda_j))} c_j \left| \frac{f(\lambda_j) - f(t\lambda_j)}{\lambda_j} \right| \leq \begin{cases} \frac{\alpha}{2}(1 - t)^\beta, & \zeta \in P, \\ \frac{\gamma}{2}(1 - t), & \zeta \in \mathbb{T} \setminus P. \end{cases}$$

Since $c_j |f(\lambda_j) - f(t\lambda_j)| \leq \rho |\lambda_j| (1 - t)$, it suffices to have

$$\sum_{j=1}^{m-1} e^{\gamma_t(1 - \text{Re } \varphi(\lambda_j))} \rho \leq \frac{\alpha}{2} (1 - t)^{\beta-1} \quad (6.1)$$

and

$$\sum_{j=1}^{m-1} e^{-\gamma_t \operatorname{Re} \varphi(\lambda_j)} \rho \leq \frac{\gamma}{2}. \quad (6.2)$$

Take γ_t such that equality in (6.1) holds. Then for t sufficiently close to 1 we also have inequality (6.2). Moreover,

$$\|h_t - f(t \cdot)\|_{\mathbb{D}} \rightarrow 0, \quad t \rightarrow 1.$$

Since D is weakly Runge, $h_t(\mathbb{D}) \subset \subset D$ for t close enough to 1.

To finish the proof suppose that there exists a set $P \subset \mathbb{T}$ of positive measure such that for all $\zeta \in P$ we have $\operatorname{dist}(f^*(\zeta), \partial D) > \varepsilon > 0$. Put

$$P_k := \{\zeta \in \mathbb{T} : \operatorname{dist}(f(t\zeta), \partial D) > \varepsilon \text{ for any } t \in (1 - 1/k, 1)\}, \quad k \in \mathbb{N}.$$

Then $P \subset \bigcup_{k=1}^{\infty} P_k$. Hence, for some k the set P_k is of positive measure, contradiction. \square

Corollary 6.3. *Any weak m -extremal of a bounded convex domain $D \subset \mathbb{C}^n$ is almost proper.*

Proof. Clearly, D is weakly Runge and further it suffices to use the Hopf lemma in the unit disc: if u is a negative subharmonic function on \mathbb{D} , then $u(\lambda) \leq -\gamma(1 - |\lambda|)$, $\lambda \in \mathbb{D}$, for some constant $\gamma > 0$.

Indeed, the function $-\operatorname{dist}(\cdot, \partial D)$ is convex on D , therefore any analytic disc $f : \mathbb{D} \rightarrow D$ satisfies $-\operatorname{dist}(f(\lambda), \partial D) \leq -\gamma(1 - |\lambda|)$ (γ depends on f). \square

Recall that a domain $\Omega \subset \mathbb{R}^m$ is said to be *strictly convex* if

$$a, b \in \overline{\Omega}, \quad a \neq b, \quad t \in (0, 1) \implies ta + (1 - t)b \in \Omega.$$

Note that a bounded domain $\Omega \subset \mathbb{R}^m$ is strictly convex if and only if

$$a, b, \frac{1}{2}(a + b) \in \partial\Omega \implies a = b.$$

Corollary 6.4 (cf. [13], Proposition 11.3.3). *Let $D \subset \mathbb{C}^n$ be a bounded strictly convex domain and let $f, g : \mathbb{D} \rightarrow D$ be weak m -extremals for $\lambda_1, \dots, \lambda_m$. Assume that $f(\lambda_j) = g(\lambda_j)$, $j = 1, \dots, m$. Then $f = g$.*

Proof. The map $h := \frac{1}{2}(f + g) : \mathbb{D} \rightarrow D$ is a weak m -extremal for $\lambda_1, \dots, \lambda_m$, whence h is almost proper. As $h^* = \frac{1}{2}(f^* + g^*)$ almost everywhere on \mathbb{T} , it follows that $f^* = g^*$ almost everywhere and $f = g$. \square

Remark 6.5. In case of the ball we can get Corollary 6.4 by induction. In fact, for $m = 2$ it is the classical result. Step $m \implies m + 1$: one may assume that $\lambda_{m+1} = 0$ and $f(0) = g(0) = 0$. Then $f(\lambda) = \lambda\varphi(\lambda)$ and $g(\lambda) = \lambda\psi(\lambda)$, where φ, ψ are either m -extremals of \mathbb{B}_n or constants lying in $\partial\mathbb{B}_n$. As $\varphi(\lambda_j) = \psi(\lambda_j)$, $j = 1, \dots, m$, the claim follows.

On the other side, in any complex ellipsoid, equality on $m - 1$ points does not suffice to claim that $f = g$. The examples are m -geodesics $f := (B, 0, \dots, 0) = -g$, where B is a Blaschke product of degree $m - 1$, having all zeros distinct.

Remark 6.6. Recall that for 2-geodesics f, g of a convex complex ellipsoid, the condition $f(\lambda_j) = g(\mu_j)$, $j = 1, 2$, where $\lambda_1, \lambda_2 \in \mathbb{D}$ are distinct and $\mu_1, \mu_2 \in \mathbb{D}$ are distinct, implies that $f = g \circ a$ for some $a \in \operatorname{Aut}(\mathbb{D})$, see [13, Proposition 16.2.2].

For $m \geq 3$ there is no an analogous property. Indeed, consider 3-geodesics $f(\lambda) := (\lambda m_\alpha(\lambda), 0, \dots, 0)$ and $g(\lambda) := (\lambda m_\beta(\lambda), 0, \dots, 0)$, where $\alpha, \beta \in \mathbb{D}$, $\alpha \neq \beta, -\beta$. Then for any $\lambda \in \mathbb{D}$ there is $\mu \in \mathbb{D}$ such that $f(\lambda) = g(\mu)$, however there is no $a \in \operatorname{Aut}(\mathbb{D})$ satisfying $f = g \circ a$ (clearly, the mappings f and g are not equivalent in case of the ball).

More generally, for any finite non-constant Blaschke products B, \tilde{B} there are infinite sets of different λ 's and μ 's with $(B(\lambda), 0, \dots, 0) = (\tilde{B}(\mu), 0, \dots, 0)$. Although, it may happen that there is no Blaschke product B_1 with $B = \tilde{B} \circ B_1$ or $\tilde{B} = B \circ B_1$, e.g. if $\deg B$ does not divide $\deg \tilde{B}$ and vice versa (moreover, $(B, 0, \dots, 0)$ and $(\tilde{B}, 0, \dots, 0)$ are not equivalent in the ball).

We pass to problems concerning properness.

Remark 6.7. (a) Any weak m -extremal of a non-simply connected taut planar domain is neither proper nor almost proper. It follows from Proposition 2.5, infiniteness of the covering and the identity principle.
(b) Any m -geodesic is proper.

We do not know whether any m -extremal is (almost) proper (P12).

Natural is the question about behavior of (weak) m -extremals and m -geodesics under compositions with proper holomorphic maps (with both sides). The problem trivializes in two cases. Indeed, if f is an m -geodesic and B is a finite non-constant Blaschke product, then $f \circ B$ is some k -geodesic. Note also that the mapping

$$\mathbb{C} \setminus \{0, 1\} \ni \lambda \mapsto \frac{1}{\lambda(\lambda - 1)} \in \mathbb{C}_*$$

is proper, but $\mathbb{C} \setminus \{0, 1\}$ has weak m -extremals, whereas \mathbb{C}_* not.

We have two simple results (cf. (P13) and (P14)).

Proposition 6.8. Let $D \subset \mathbb{C}^n$ be a convex domain and let $f : \mathbb{D} \rightarrow D$ be an m -extremal. Assume that B is a Blaschke product of degree $k \in \mathbb{N}$. Then $f \circ B : \mathbb{D} \rightarrow D$ is a weak mk -extremal.

Proof. Let $M := \{\lambda \in \mathbb{D} : B'(\lambda) = 0\}$ and let $\mu_1, \dots, \mu_m \in \mathbb{D} \setminus B(M)$ be different. We will show that $f \circ B$ is a weak mk -extremal for elements of the set $\Lambda := B^{-1}(\{\mu_1, \dots, \mu_m\})$ (the structure of proper holomorphic mappings is used, cf. [4] and [29, Chapter 15]). Suppose that there exists $h \in \mathcal{O}(\mathbb{D}, D)$ such that $h(\lambda) = f(B(\lambda))$, $\lambda \in \Lambda$, and $h(\mathbb{D}) \subset\subset D$. For any $\mu \in \mathbb{D} \setminus B(M)$ let $B_{\mu,1}, \dots, B_{\mu,k}$ denote the local inverses of B in a neighborhood U_μ of μ . Then

$$\frac{1}{k}(h \circ B_{\mu,1} + \dots + h \circ B_{\mu,k}) = \frac{1}{k}(h \circ B_{\nu,1} + \dots + h \circ B_{\nu,k}) \text{ on } U_\mu \cap U_\nu$$

for $\mu, \nu \in \mathbb{D} \setminus B(M)$. We glue these mappings to $g \in \mathcal{O}(\mathbb{D} \setminus B(M), D)$. Then $g(\mu_j) = f(\mu_j)$ for any j and $g(\mathbb{D} \setminus B(M)) \subset\subset D$. Clearly, g extends holomorphically to \mathbb{D} and the extension has a relatively compact image, contradiction. \square

Remark 6.9. The property of being some (weak) m -extremal (resp. m -geodesic) is not invariant under proper holomorphic mappings in different dimensions. Indeed, there exists a function u harmonic in \mathbb{D} , continuous to the boundary and such that its harmonic conjugate v is not continuous on $\overline{\mathbb{D}}$. We give an example from [33, p. 253]

$$u(e^{it}) := \sum_{j=2}^{\infty} \frac{\sin jt}{j \log j}, \quad t \in \mathbb{R}.$$

Adding a constant, we can assume that $u < 0$ in $\overline{\mathbb{D}}$. Define $\tilde{u} := 1/2 \log(1 - e^{2u})$ on \mathbb{T} , extend it harmonically to \mathbb{D} and take \tilde{v} as its harmonic conjugate. The map $\Phi := (e^{u+iv}, e^{\tilde{u}+i\tilde{v}}) : \mathbb{D} \rightarrow \mathbb{B}_2$ is proper, but $\Phi \circ \text{id}_{\mathbb{D}}$ does not extend to $\overline{\mathbb{D}}$, so it is not any weak m -extremal of \mathbb{B}_2 .

Following the proof of [8, Proposition 9] we get the last result.

Proposition 6.10 (cf. [8], Proposition 9). *Let $D \subset \mathbb{C}^n$ be a domain and let $f : \mathbb{D} \rightarrow D$ be a holomorphic mapping such that for some $\gamma > 0$ we have*

$$\text{dist}(f(\lambda), \partial D) \geq \gamma(1 - |\lambda|), \quad \lambda \in \mathbb{D}. \quad (6.3)$$

Assume that f is a weak m -extremal for $\lambda_1, \dots, \lambda_m$. Then $f'(\lambda_j) \neq 0$ for at least two j 's.

Proof. Suppose contrary, say $f'(\lambda_j) = 0$, $j = 1, \dots, m-1$. Then $g := f \circ m_{-\lambda_m}$ is a weak m -extremal for some $\mu_1, \dots, \mu_{m-1}, 0$ and $g'(\mu_j) = 0$ for any $1 \leq j \leq m-1$. Moreover, condition (6.3) for g holds with possibly another constant.

For $t \in (0, 1)$ consider the mapping

$$h_t(\lambda) := g(t\lambda) + \sum_{j=1}^{m-1} \left(\frac{\lambda}{\mu_j} \prod_{k \neq j} \frac{\lambda - \mu_k}{\mu_j - \mu_k} \right) (g(\mu_j) - g(t\mu_j)), \quad \lambda \in \mathbb{D}.$$

Then h_t interpolates g at $\mu_1, \dots, \mu_{m-1}, 0$ and $\|\psi_t\|_{\mathbb{D}} \rightarrow 0$ as $t \rightarrow 1$, where

$$\psi_t(\lambda) := \frac{h_t(\lambda) - g(t\lambda)}{1 - t}.$$

Hence, for t sufficiently close to 1 we have $h_t(\mathbb{D}) \subset \subset D$. □

7. LIST OF PROBLEMS

- (P1) Does there exist a 2-extremal, which is not a 2-geodesic?
- (P2) Does there exist an m -extremal being not any k -geodesic?
- (P3) Let $D \subset \mathbb{C}^n$ be a k -balanced pseudoconvex domain and let $f : \mathbb{D} \rightarrow D$ be an m -extremal. Assume that $k_1, \dots, k_n \leq 1$. Decide whether the mapping $\psi(\lambda) := (\lambda^{k_1} f_1(\lambda), \dots, \lambda^{k_n} f_n(\lambda))$ is an $(m+1)$ -extremal.
- (P4) Let $f : \mathbb{D} \rightarrow \mathcal{E}(p)$ be a 4-geodesic such that $f(\lambda) = \lambda\varphi(\lambda)$, $\varphi \in \mathcal{O}(\mathbb{D}, \mathcal{E}(p))$. Does it follow that φ is a 3-geodesic?
- (P5) Is any weak m -extremal of a convex domain an m -extremal?
- (P6) Decide whether any map of the form (4.4) is some (weak) l -extremal or l -geodesic.
- (P7) Does weak m -extremality coincide with m -extremality in any convex complex ellipsoid?
- (P8) Decide whether any non-constant map $(a_1 B_1, \dots, a_n B_n)$ ($a \in \partial \mathcal{E}(p)$, B_j 's finite Blaschke products) is some (weak) m -extremal or m -geodesic.
- (P9) Is any m -extremal of \mathbb{B}_n equivalent with some $(a_1 B_1, \dots, a_n B_n)$?
- (P10) Is any m -extremal of \mathbb{B}_n some k -geodesic?
- (P11) Let $0 < a < 1$. Does it follow that the mapping $f(\lambda) := (a\lambda, \sqrt{1-a^2}\lambda^m)$ is an $(m+1)$ -geodesic of \mathbb{B}_2 ?
- (P12) Decide whether any m -extremal is (almost) proper.
- (P13) Let f be a (weak) m -extremal and B a finite non-constant Blaschke product. Does it follow that $f \circ B$ is some (weak) k -extremal?
- (P14) Is the property of being some m -extremal (resp. m -geodesic) invariant under proper holomorphic mappings in the same dimension?

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